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Average Error for Spectral Asymptotics on Surfaces

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Abstract

Let $N(t)$ denote the eigenvalue counting function of the Laplacian on a compact surface of constant nonnegative curvature, with or without boundary. We define a refined asymptotic formula $\widetilde{N}(t) = At + Bt^{1/2} + C$, where the constants are expressed in terms of the geometry of the surface and its boundary, and consider the average error $A(t) = \frac{1}{t}\int_0^t D(s)\,ds$ for $D(t) = N(t) - \widetilde{N}(t)$. We present a conjecture for the asymptotic behavior of $A(t)$, and study some examples that support the conjecture.

"The mills of God grind slowly, yet they grind exceeding small."

Proverb

1 Introduction

For any positive self-adjoint operator with discrete spectrum

$$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty \qquad \text{(repeated according to multiplicity)}$$

we consider the eigenvalue counting function

$$N(t) = \#\{\lambda_j \le t\}. \tag{1.1}$$

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Often there is a predicted asymptotic approximation $\widetilde{N}(t)$, and the problem arises to estimate the error

$$D(t) = N(t) - \widetilde{N}(t). \tag{1.2}$$

Here we investigate some examples where the average error

$$A(t) = \frac{1}{t}\int_0^t D(s)\,ds \tag{1.3}$$

is better behaved, following up on [JS]. In our examples we deal with $-\Delta$ on a compact surface S which is either flat or has constant positive curvature. The Weyl asymptotic formula has $N(t) = At + \mathrm{o}(t)$ where the constant A is given by $A = \frac{1}{4\pi}\mathrm{Area}(S)$. If S has a reasonable boundary then there is a second term in the asymptotics, $N(t) = At + Bt^{1/2} + \mathrm{o}(t^{1/2})$, where the constant B depends on the boundary conditions. If we impose Neumann boundary conditions on all of ∂S, then $B = \frac{1}{4\pi}\mathrm{length}(\partial S)$, and if we impose Dirichlet boundary conditions then $B = -\frac{1}{4\pi}\mathrm{length}(\partial S)$. We may also consider mixed boundary conditions, splitting $\partial S = \partial S_N \cup \partial S_D$ and imposing Neumann boundary conditions on ∂S_N and Dirichlet boundary conditions on ∂S_D. In that case $B = \frac{1}{4\pi}\mathrm{length}(\partial S_N) - \frac{1}{4\pi}\mathrm{length}(\partial S_D)$.

For our purposes we will need a more refined asymptotics with an additional constant term:

$$\widetilde{N}(t) = At + Bt^{1/2} + C \tag{1.4}$$

where A and B are as above and C is given as follows. We assume that ∂S is piecewise smooth, with a finite number of corners with angles $\{\theta_j\}$. We will write $C = C_1 + C_2 + C_3$, where C_1 is the contribution from the corners, C_2 is the contribution from the curvature of the smooth arcs in ∂S, and C_3 is the contribution from the curvature of S. Thus C_3 will be zero if S is flat, and

$$C_3 = \frac{1}{12\pi}K_2(S), \tag{1.5}$$

where $K_2(S)$ is the total curvature, the integral of the curvature over S, in the general case. Thus for any sphere $K(S) = 4\pi$ and $C_3 = \frac{1}{3}$. For C_2 we will take the integral of the curvature K_1 of the boundary,

$$C_2 = \frac{1}{12\pi}\int_{\partial S} K_1. \tag{1.6}$$

Here the curvature is taken with respect to S, so it will be multiplied by -1 on the interior portions of the boundary. If ∂S has no corners and consists of an outer boundary and N inner boundary curves, then $C_2 = \frac{1-N}{6}$.

To describe the constant C_1 we define

$$\psi(\theta) = \frac{1}{24}\left(\frac{\pi}{\theta} - \frac{\theta}{\pi}\right). \tag{1.7}$$

If we impose Neumann or Dirichlet conditions throughout, then we take

$$C_1 = \sum \psi(\theta_j) \tag{1.8}$$

where the sum is over all corner points: For mixed boundary condition we subdivide the boundary at corner points, and impose Neumann conditions on the arcs between some corner points, and Dirichlet conditions on the remaining arcs. We then sort the corner points into those $\{\theta'_j\}$ where the same boundary condition is imposed on both incident arcs, and $\{\theta''_j\}$ where different boundary conditions are imposed. Then we take

$$C_1 = \sum \psi(\theta'_j) + \sum(\psi(2\theta''_j) - \psi(\theta''_j)). \tag{1.9}$$

We may also allow the surface to have a finite number of cone point singularities with cone angles $\{\alpha_j\}$, $0 < \alpha_j < 2\pi$. In that case we add to C_1 the value

$$\sum 2\psi_j\left(\frac{\alpha_j}{2}\right). \tag{1.10}$$

Conjecture. *(a) Suppose S is flat. Then there exists a uniformly almost periodic function g of mean value zero, such that*

$$A(t) = g(t^{1/2})t^{-1/4} + O(t^{-1/2}), \tag{1.11}$$

and more generally there exists a sequence $\{g_j\}$ of uniformly almost periodic functions, with $g_1 = g$, such that for all N

$$A(t) = \sum_{j=1}^{N} g_j(t^{1/2})t^{-j/4} + O(t^{-(N+1)/4}) \tag{1.12}$$

(b) Suppose S has constant positive curvature. Then there exists a uniformly almost periodic function g of mean value zero, such that

$$A(t) = g(\sqrt{t+\frac{1}{4}}) + O(t^{-1/2}), \tag{1.13}$$

and more generally there exists a sequence $\{g_j\}$ of uniformly almost periodic functions, with $g_1 = g$, such that for all N

$$A(t) = \sum_{j=1}^{N} g_j(\sqrt{t + \frac{1}{4}})t^{-(j-1)/2} + O(t^{-N/2}) \tag{1.14}$$

In this paper we present a number of examples that support the conjecture. Of course this is exactly backwards, since the examples were worked out first and the conjecture was concocted to agree with the examples. All of the examples are sufaces for which it is possible to compute the spectrum of the Laplacian exactly in terms of trigonometric polynomials for flat surfaces and spherical harmonics for positively curved surfaces. In particular, they are highly symmetric. So they provide only weak evidence for the conjecture, and the methods of this paper do not provide a pathway to attack the conjecture. Nevertheless, the conjecture seems interesting enough that it is worth submitting to the mathematical community to stimulate further work. We should also point out that although all the almost periodic functions in our examples of positively curved surfaces are in fact periodic, we have not made periodicity part of conjecture (b).

For surefaces without boundary the following relationship appears to hold: the set of frequencies in the trigonometric expansions of the almost periodic functions coincides with the set of lengths of closed geodesics on the surface. At present we have no explanation for why this should be the case.

There should be an analogous conjecture for surfaces of constant negative curvature. The expression for the refined asymptotics $\widetilde{N}(t)$ should be the same, but it is not clear what should replace (1.11) and (1.13). There is a vast literature on the spectrum of the Laplacian on compact Riemann surfaces, or even surfaces of finite volume, and the relationship with the lengths of closed geodesics, going back to the Selberg trace formula (see [Bu], [Sa] for example). Although the spectral projection operator on the hyperbolic plane is known explicitly ([Ta], [S]), there is apparently no known exact computation on any quotient space. Since we have no examples to compute, we will not venture a conjecture.

There is also a vast literature on the asymptotics of the trace of the heat kernel ([G]). The trace of the heat kernel is given as a smoothing of the

eigenvalue counting function

$$h(t) = \sum e^{t\lambda_j} = t\int_0^\infty N(s)e^{-st}ds, \tag{1.15}$$

so asymptotics of $N(t)$ translate immediately into asymptotics of the heat kernel trace. If we substitute $N(s) - \widetilde{N}(s) = \frac{d}{ds}(sA(s))$ in (1.15) and integrate by parts we obtain

$$h(t) - \widetilde{h}(t) = t^2\int_0^\infty A(s)se^{-st}\,ds \qquad \text{where} \tag{1.16}$$

$$\widetilde{h}(t) = t\int_0^\infty \widetilde{N}(s)e^{-st}\,ds, \tag{1.17}$$

so asymptotics for $A(t)$ also translate into asymptotics of $h(t)$. To go in the reverse direction requires the application of a Tauberian Theorem, and cannot reveal the refined asymptotic statements that we are interested in. However, we can check that the refined asymptotics $\widetilde{N}(t)$ that we use are consistent with the known asymptotics for $h(t)$ ([BS], [K], [G]). Our average function $A(t)$ is a much cruder smoothing of $N(t)$ than the trace of the heat kernel. This is a double edged sword. On the one hand, we can't expect as nice behavior. On the other hand, the rougher behavior reveals some interesting new features; for example, the almost periodic functions.

If the boundary is smooth, we observed that the contribution C_2 to the constant term is given by "topological data," namely $C_2 = \frac{1-g}{6}$ where g is the genus of the surface. The analogous statement for the trace of the heat kernel is the grand finale to the famous paper of Mark Kac [K]. It is interesting to note that Kac's argument is based on approximating the smooth boundary by a polynomial curve, and he is able to say something about the polynomial case in terms of some frightful integrals, but only in the limit does the result become comprehensible. (An explicit formula for the polynomial case is given in [BS].)

In our approach, the polygonal case is quite explicit, but we may also check that there is continuity in approximating the smooth boundary by a polygonal boundary. To simplify the discussion, assume that the surface is a simply connected convex planar domain and has a smooth outer boundary ∂S. Subdivide the boundary at n points $\{x_j\}$, and connect them by line

segments to form a polygon P. Then the constant term in $\widetilde{N}$ for the polygonal domain is entirely given by C_1, namely (1.8) where θ_j is the angle at x_j. As n gets large the values of θ_j approach π from below, and

$$24\psi(\theta_j) = 2(1 - \frac{\theta_j}{\pi}) + O((1 - \frac{\theta_j}{\pi})^2). \tag{1.18}$$

A simple geometric argument shows

$$\sum \theta_j = (n-2)\pi \tag{1.19}$$

and so

$$C_1 = \sum \psi(\theta_j) = \frac{1}{24}(2n - \frac{2}{\pi}\sum \theta_j) + R = \frac{1}{6} + R \tag{1.20}$$

where the remainder R is $O(\sum(1 - \frac{\theta_j}{\pi})^2)$ and tends to zero in the limit. Thus the constant term for the polygonal approximations approximates the constant term (C_2) for the smooth domain. It is straightforward to localize this argument to polygonal approximations to portions of the boundary of any surface.

We want to emphasize that it is important to average the error $D(t)$ in order to get the asymptotic behavior. In all our examples the function $D(t)$ is unbounded. It is the fact that it is positive and negative that allows cancellation to produce bounded behavior for $A(t)$. Compared with the values of $D(t)$, the constant term in the asymptotics $\widetilde{N}(t)$ is indeed "exceeding small." The fact that it nevertheless shows up in the average seems truly remarkable.

Similar results should be valid in higher dimensions, but they will require a different type of average. We leave this to the future.

We now outline the examples that take up the rest of this paper. There are two basic examples, the torus discussed in [JS] and the sphere, discussed in section 4. In all the other examples, the eigenfunctions on the surface may be extended to eigenfunctions on a torus or a sphere, so the spectrum on the surface is a subset of the spectrum for one of our basic examples. Our task is then to identify exactly the subset, and show how the conjecture for the basic example yields the conjecture for the surface. This requires only elementary reasoning, but the arguments are a bit technical. Of course we have to be very careful, since the value of the constant depends on getting

exact statements. It may seem that we are working out a lot of examples using very similar arguments. However, we found that we needed all the examples to help formulate and confirm the conjecture. We have tried to present enough detail so that the reader can verify the correctness of the result, without excessively repeating the framework of the reasoning. We have used what we hope is self explanatory notation, but it changes from example to example. So, for example, we write $N(t)$ for the counting function for the surface under discussion (or $N_N(t)$ or $N_D(t)$ if there is a boundary with Neumann or Dirichlet boundary conditions). If we have to recall a counting function from a previous example we will write $N_T(t)$ for a torus T, etc.

In section 2 we discuss examples that are polygons in the plane with either Neumann or Dirichlet boundary conditions throughout. These examples are arbitrary rectangles, and certain special triangles: right isosceles, equilateral, and $30° - 60° - 90°$.

In section 3 we return to the same surfaces, but deal with mixed boundary conditions, Neumann on part of the boundary and Dirichlet on the remainder. We are able to handle all possibilities for the rectangle and right isosceles triangle, but none for the equilateral triangle and only some for the $30° - 60° - 90°$ triangle. In this section we also consider an arbitrary cylinder and Möbius band.

In section 4 we consider the sphere, the hemisphere and the projective sphere. In section 5 we consider lunes and half-lunes. These examples are very useful because they provide a wider variety of corner angles than the previous examples.

In section 6 we discuss surfaces with point singularities. These include the flat projective plane discussed in [JS], the surface of a regular tetrahedron discussed in [GKS], half tetrahedra, and glued-lunes.

In section 7 we re-examine some of our examples of surfaces that have a finite group G of isometries. The question is how the spectrum sorts into the eigenfunctions with prescribed symmetry, given by the irreducible representations $\{\pi_j\}$ of G. A heuristic suggested in [S] is that the proportions $N_j(t)/N(t)$, where $N_t(t)$ is the counting function for eigenfunctions of π_j symmetry types is asymptotically $(\dim \pi_j)^2/\#G$. Here we work out refined

asymptotics $\widetilde{N}_j(t)$ for $N_j(t)$, by reducing the computation of $N_j(t)$ to previously worked out examples for a fundamental domain of the G action. The leading term At is as predicted, but the next term $Bt^{1/2}$ may be positive, negative, or zero, showing that the individual representations are somewhat overrepresented or underepresented in the spectrum. Because we only have a few examples, we are not able to formulate a conjecture for the behavior in the general case. This is another interesting open problem for the future.

2 Polygons

Consider the rectangle R of side length, a, b, and let T denote the torus of side length $2a, 2b$. Any Neumann eigenfunction on R extends by even reflection to an eigenfunction on T, and similarly a Dirichlet eigenfunction extends by odd reflection. In either case we obtain roughly a quarter of the eigenfunction on T, but in fact we can give a precise formula relating the counting functions N_N and N_D for R with N_T for T.

The eigenfunctions on T have the form

$$e(j,k) = e^{\pi i(\frac{j}{a}x+\frac{k}{b}y)}, \qquad j,k \in \mathbb{Z}, \tag{2.1}$$

with eigenvalue $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2}$. The Neumann eigenfunctions on R have the form

$$c(j,k) = \cos \pi \frac{j}{a} x \cos \pi \frac{k}{b} y, \qquad j \geq 0,\ k \geq 0, \tag{2.2}$$

while the Dirichlet eigenfunctions have the form

$$s(j,k) = \sin \pi \frac{j}{a} x \sin \pi \frac{k}{b} y, \qquad j > 0,\ k > 0, \tag{2.3}$$

with the same eigenvalue.

Lemma 2.1. *We have*

$$N_N(t) = \frac{1}{4} N_T(t) + \frac{1}{2}\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\left[\frac{bt^{1/2}}{\pi}\right] + \frac{3}{4} \tag{2.4}$$

$$N_D(t) = \frac{1}{4} N_T(t) - \frac{1}{2}\left[\frac{at^{1/2}}{\pi}\right] - \frac{1}{2}\left[\frac{bt^{1/2}}{\pi}\right] - \frac{1}{4} \tag{2.5}$$

Proof. In the generic case $j > 0$ and $k > 0$, we have four eigenfunctions $e(\pm j, \pm k)$ contributing to N_T for one eigenfunction $c(j,k)$ or $s(j,k)$ contributing to N_N or N_D, giving rise to the $\frac{1}{4}N_T(t)$ terms in (2.4) and (2.5). We then have to correct for the nongeneric cases. Note that $e(0,0)$ contributes once to N_T and N_N but not to N_D, so this gives rise to the constant terms. When $k = 0$ and $j > 0$, we have $e(\pm j, 0)$ for $\frac{\pi^2 j^2}{a^2} \leq t$ contributing to $N_T(t)$ and $c(j,0)$ contributing to $N_N(t)$ but not $N_D(t)$, so this gives rise to the term $\pm\frac{1}{2}\left[\frac{at^{1/2}}{\pi}\right]$ in (2.4) and (2.5). Similarly the case $k > 0$ and $j = 0$ gives rise to the term $\pm\frac{1}{2}\left[\frac{bt^{1/2}}{\pi}\right]$. □

Note that the function $[x]$ is on average $x - \frac{1}{2}$, so we may rewrite (2.4) and (2.5) as

$$N_N(t) = \frac{1}{4}N_T(t) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{bt^{1/2}}{\pi}\right] + \frac{1}{2}\right) + \frac{1}{4} \tag{2.6}$$

$$N_D(t) = \frac{1}{4}N_T(t) - \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{bt^{1/2}}{\pi}\right] + \frac{1}{2}\right) + \frac{1}{4} \tag{2.7}$$

Now the constant term is the same in both equations. We define the refined asymptotics

$$\widetilde{N}_N(t) = \frac{ab}{4\pi}t + \frac{2a+2b}{4\pi}t^{1/2} + \frac{1}{4} \tag{2.8}$$

$$\widetilde{N}_D(t) = \frac{ab}{4\pi}t - \frac{2a+2b}{4\pi}t^{1/2} + \frac{1}{4}. \tag{2.9}$$

Note that ab is the area of R and $2a + 2b$ is the length of the perimeter of R as predicted, and $\frac{1}{4} = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$ as predicted. As usual we define the discrepancy $D(t)$ and average $A(t)$ in each of the three cases.

Theorem 2.2. *Both A_N and A_D satisfy*

$$A(t) = g(t^{1/2})t^{-1/4} + O(t^{-1/2}) \quad \textit{as } t \to \infty \tag{2.10}$$

where g is an almost periodic function of mean value zero.

Proof. We have

$$\left\{ \begin{array}{c} D_N(t) \\ D_D(t) \end{array} \right\} = \pm\frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2} - \frac{at^{1/2}}{\pi}\right) \pm \frac{1}{2}\left(\left[\frac{bt^{1/2}}{\pi}\right] + \frac{1}{2} - \frac{bt^{1/2}}{\pi}\right) + \frac{1}{4}D_T(t). \tag{2.11}$$

Since (2.10) holds for A_T by Theorem 4 of [JS], it suffices to show that an estimate like (2.10) holds for the function $[ct^{1/2}] + \frac{1}{2} - ct^{1/2}$. For simplicity take $c = 1$. We need to estimate

$$\begin{aligned} &\frac{1}{t}\int_0^t ([s^{1/2}] + \frac{1}{2} - s^{1/2})\, ds = \frac{2}{t}\int_0^{\sqrt{t}} ([r] + \frac{1}{2} - r) r\, dr \\ &= \frac{2}{t}\sum_{k=1}^{[t^{1/2}]} \int_{k-1}^{k} (k - \frac{1}{2} - r)\, r\, dr \; + \frac{2}{t}\int_{[t^{1/2}]}^{t^{1/2}} ([t^{1/2}] + \frac{1}{2} - r)\, r\, dr. \end{aligned} \tag{2.12}$$

Now $\int_{k-1}^{k} (k - \frac{1}{2} - r)\, r\, dr = -\frac{1}{4}$, so the first term in (2.12) is exactly $\frac{-[t^{1/2}]}{2t} = O(t^{-1/2})$. For the second term we note that the integrand is $O(t^{1/2})$ and the interval of integration has length at most 1, so again the contribution is $O(t^{-1/2})$. □

The almost periodic function g is the same for A_N and A_D, and aside from the factor 1/4 it is given explicitly in [JS]. The error estimate $O(t^{-1/2})$ in (2.10) is somewhat worse than the estimate $O(t^{-3/4})$ for A_T given in [JS]. We may also regard (2.10) as the first term in an asymptotic expansion with $O(t^{-1/2})$ replaced by sums of $g_k(t^{1/2})t^{-k/4}$. For odd values of k the g_k are almost periodic functions as given in [JS] arising from $\frac{1}{2}A_T$. The expression (2.12) may be written as $g_2(t^{1/2})t^{-1/2} + g_4(t^{1/2})t^{-1}$ where g_2 and g_4 are periodic of period 1. This is easily seen because $h(x) = x - [x]$ is such a periodic function, and (2.12) is a polynomial in $h(t^{1/2})$ and $t^{1/2}$ divided by t.

Next we consider a right isosceles triangle that is half of a square of side length a. Then Neumann and Dirichlet eigenfunctions extend by even and odd reflection to eigenfunctions of the same type. Continuing the same notation as before with $b = a$, the Neumann eigenfunctions are

$$c(j,k) + c(k,j) \qquad \text{for } 0 \le j \le k \tag{2.13}$$

and the Dirichlet eigenfunctions are

$$s(j,k) - s(k,j) \qquad \text{for } 0 < j < k. \tag{2.14}$$

Lemma 2.3. *For the right isosceles triangle we have*

$$N_N(t) = \frac{1}{8}N_T(t) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\sqrt{2}\pi}\right] + \frac{1}{2}\right) + \frac{3}{8} \tag{2.15}$$

$$N_D(t) = \frac{1}{8}N_T(t) - \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{at^{1/2}}{\sqrt{2}\pi}\right] + \frac{1}{2}\right) + \frac{3}{8} \tag{2.16}$$

Proof. In the generic case $0 < j < k$ there are eight eigenfunctions $e(\pm j, \pm k)$ and $e(\pm k, \pm j)$ contributing to N_T for one eigenfunction (2.13) or (2.14) contributing to N_N or N_D. The case $(j,k) = (0,0)$ contributes the constant. The case $k > j = 0$ contributes $e(\pm k, 0)$ and $e(0, \pm k)$ to N_T, but only $c(k,0) + c(0,k)$ to N_N and nothing to N_D, leading to the term $\frac{1}{2}\left[\frac{at^{1/2}}{\pi}\right]$ in (2.15) and (2.16). When $j = k > 0$ we have the eigenvalue $\frac{2\pi^2 k^2}{a^2}$, with $e(\pm k, \pm k)$ contributing to N_T, only $c(k,k)$ contributing to N_D, and no contribution to N_N. This leads to the term $\frac{1}{2}\left[\frac{at^{1/2}}{\sqrt{2}\pi}\right]$ in (2.15) and (2.16). □

Thus we define the refined asymptotics

$$\widetilde{N}_N(t) = \frac{a^2}{8\pi}t + \left(\frac{2+\sqrt{2}}{4\pi}\right)at^{1/2} + \frac{3}{8}, \tag{2.17}$$

$$\widetilde{N}_D(t) = \frac{a^2}{8\pi}t - \left(\frac{2+\sqrt{2}}{4\pi}\right)at^{1/2} + \frac{3}{8}, \tag{2.18}$$

Theorem 2.4. *For the right isosceles triangle we have the estimate (2.10) for A_N and A_D*

Proof. Same as for Theorem 2.2. □

Note that $\frac{3}{8} = \frac{1}{16} + \frac{5}{32} + \frac{5}{32}$ as predicted for the angles $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$.

Next we consider the equilateral triangle. For simplicity we assume the side length is 1. Six equilateral triangles tile a regular hexagon, (see figure 2.1), and Dirichlet and Neumann eigenfunctions extended by even or odd reflection extend to periodic functions on the plane with respect to the lattice associated to the hexagonal torus T.
We may choose a basis $(\sqrt{3}, 0)$ and $(\frac{\sqrt{3}}{2}, \frac{3}{2})$ for the lattice $\mathscr{L}$, and a basis $u = (\frac{1}{\sqrt{3}}, \frac{1}{3})$ and $v = (0, \frac{2}{3})$ for the dual lattice $\mathscr{L}^*$. Then the torus eigenfunctions are of the form

$$\widetilde{e}(k,j) = e^{2\pi(ku+jv)\cdot x} \qquad k, j \in \mathbb{Z} \tag{2.19}$$

with eigenvalue $\left(\frac{4\pi}{3}\right)^2 (k^2 + j^2 + kj)$.

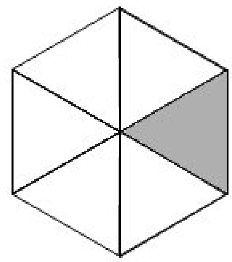


Figure 2.1

To describe the Neumann and Dirichlet eigenfunctions on the triangle it is convenient to think of the dual lattice $\mathscr{L}^*$ as made up of the origin surrounded by concentric hexagons. In Figure 2.2 we show one such hexagon with a generic choice ($k > j > 0$) of twelve points associated to the same eigenvalue, together with three reflection axes of the triangle sides.

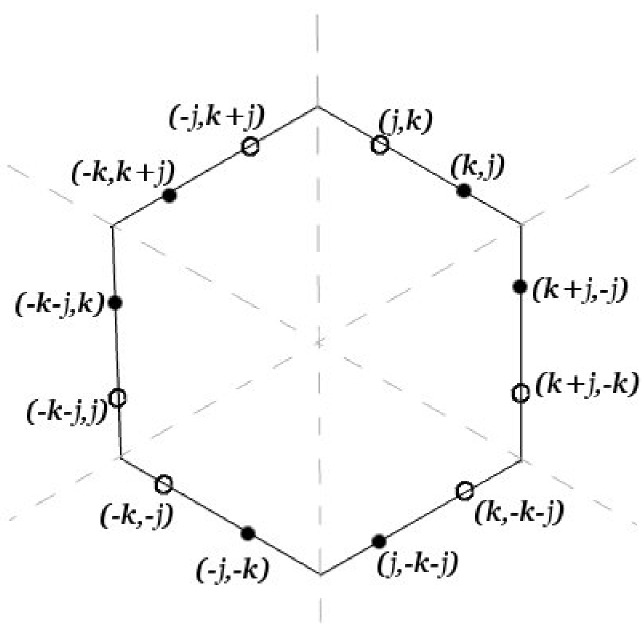


Figure 2.2

A Neumann eigenfunction must be symmetric with respect to the three re-

flections, so there are two eigenfunctions associated to these lattice points, namely

$$\begin{aligned} &\widetilde{e}(k,j)+\widetilde{e}(-k,k+j)+\widetilde{e}(-k-j,k)+\widetilde{e}(-j,-k) \\ &\qquad\qquad\qquad +\widetilde{e}(j,-k-j)+\widetilde{e}(k+j,-j) \end{aligned} \tag{2.20}$$

(dark points) and the same with j and k interchanged (open points). Similarly a Dirichlet eigenfunction must be skew-symmetric, so again we find two,

$$\begin{aligned} &\widetilde{e}(k,j)-\widetilde{e}(-k,k+j)+\widetilde{e}(-k-j,k)-\widetilde{e}(-j,-k) \\ &\qquad\qquad\qquad +\widetilde{e}(j,-k-j)-\widetilde{e}(k+j,-j) \end{aligned} \tag{2.21}$$

and the same with j and k interchanged. For the nongeneric cases we have $(0,0)$ contributing to N_T and N_N but not N_D, (k,k) contributing six terms to N_T and one each to N_N and N_D, and $(k,0)$ contributing six terms to N_T and two terms to N_N, namely $\widetilde{e}(k,0)+\widetilde{e}(-k,k)+\widetilde{e}(0,-k)$ and $\widetilde{e}(-k,0)+\widetilde{e}(k,-k)+\widetilde{e}(0,k)$ and nothing to N_D. This yields

$$N_N(t)=\frac{1}{6}N_T(t)+\left(\left[\frac{3}{4\pi}t^{1/2}\right]+\frac{1}{2}\right)+\frac{1}{3}, \tag{2.22}$$

$$N_D(t)=\frac{1}{6}N_T(t)-\left(\left[\frac{3}{4\pi}t^{1/2}\right]+\frac{1}{2}\right)+\frac{1}{3}. \tag{2.23}$$

Thus we define the refined asymptotics

$$\widetilde{N}_N(t)=\frac{1}{4\pi}\frac{\sqrt{3}}{4}t+\frac{3}{4\pi}t^{1/2}+\frac{1}{3}, \tag{2.24}$$

$$\widetilde{N}_D(t)=\frac{1}{4\pi}\frac{\sqrt{3}}{4}t-\frac{3}{4\pi}t^{1/2}+\frac{1}{3}, \tag{2.25}$$

and the analog of Theorem 2.2 is valid. Note that $\frac{1}{3}=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}$ as predicted.

Finally, we consider the $30^\circ-60^\circ-90^\circ$ triangle that is half of the equilateral triangle. For Neumann eigenfunctions that means we take Neumann eigenfunctions on the equilateral triangle that are symmetric with respect to reflection in the x-axis, and similarly for Dirichlet eigenfunctions with skew-symmetry. For a generic choice $k>j>0$ we get exactly one eigenfunction of each type by taking the sum of the two eigenfunctions of the form (2.20)

or (2.21), with j and k interchanged. For the nongeneric cases we have $(0,0)$ contributing to N_T and N_N but not N_D, (k,k) contributing six terms to N_T, one term to N_N and nothing to N_D, and $(k,0)$ contributing six terms to N_T, one term to N_N and nothing to N_D. Thus

$$N_N(t) = \frac{1}{12}N_T(t) + \frac{1}{2}\left(\left[\frac{\sqrt{3}}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{3}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) + \frac{5}{12}, \tag{2.26}$$

$$N_D(t) = \frac{1}{12}N_T(t) - \frac{1}{2}\left(\left[\frac{\sqrt{3}}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{3}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) + \frac{5}{12}. \tag{2.27}$$

Thus the analog of Theorem 2.2 holds for

$$\widetilde{N}_N(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{8}t + \frac{3+\sqrt{3}}{8\pi}t^{1/2} + \frac{5}{12}, \tag{2.28}$$

$$\widetilde{N}_D(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{8}t - \frac{3+\sqrt{3}}{8\pi}t^{1/2} + \frac{5}{12}. \tag{2.29}$$

Note that $\frac{3+\sqrt{3}}{2}$ is the length of the perimeter of the triangle, and $\frac{5}{12} = \frac{1}{16} + \frac{1}{9} + \frac{35}{144}$ as predicted.

3 Mixed Boundary Conditions

In this section we look at examples of polygons with Neumann conditions on part of the boundary and Dirichlet conditions on the rest of the boundary.

We begin with the simplest case: a rectangle R with Neumann conditions on facing edges of length a and Dirichlet conditions on the facing edges of length b. Just as in the pure Neumann and Dirichlet example in section 2 we may extend eigenfunctions to the same torus T, and so the eigenfunctions have the form

$$f(j,k) = \cos \pi\frac{j}{a}x \sin \pi\frac{k}{b}y \qquad \text{for } j \geq 0 \text{ and } k > 0, \tag{3.1}$$

with eigenvalue $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2}$. Just as in Lemma 2.1 we have

$$\begin{aligned}N(t) &= \frac{1}{4}N_T(t) + \frac{1}{2}\left[\frac{at^{1/2}}{\pi}\right] - \frac{1}{2}\left[\frac{bt^{1/2}}{\pi}\right] - \frac{1}{4} \\ &= \frac{1}{4}N_T(t) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{bt^{1/2}}{\pi}\right] + \frac{1}{2}\right) - \frac{1}{4},\end{aligned} \tag{3.2}$$

and so we define the refined asymptotics

$$\widetilde{N}(t) = \frac{ab}{4\pi}t + \frac{2a-2b}{4\pi}t^{1/2} - \frac{1}{4}. \tag{3.3}$$

Then the analog of Theorem 2.2 holds. Note that $2a - 2b$ gives the difference of the lengths of the portions of the perimeter where Neumann and Dirichlet boundary conditions hold, and the constant is $-\frac{1}{4}$ because each of the four vertices has mixed boundary conditions on the incident edges, with $\psi(\pi) - \psi(\frac{\pi}{2}) = -\frac{1}{16}$.

Next we consider the case where the same type of boundary conditions hold on one pair of opposite edges (say the ones of length b), while for the other pair we have Neumann on one side and Dirichlet on the other. We call these the NM (Neumann/mixed) and DM (Dirichlet/mixed) cases. Here we need a larger torus T' of size $2a \times 4b$ on which to extend the eigenfunctions. The eigenfunctions then have the form

$$f(j,k) = \begin{cases} \cos \pi \frac{j}{a} x \cos \pi \left(\frac{k+\frac{1}{2}}{b}\right) y & j \geq 0, k \geq 0 \quad \text{(NM)} \\ \sin \pi \frac{j}{a} x \cos \pi \left(\frac{k+\frac{1}{2}}{b}\right) y & j > 0, k \geq 0 \quad \text{(DM)} \end{cases} \tag{3.4}$$

with eigenvalue

$$\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 (2k+1)^2}{4b^2}. \tag{3.5}$$

Now we observe that $N_{T'}(t)$ also has eigenvalues $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 (2k)^2}{4b^2}$, so $N_{T'}(t) - N_T(t)$ counts all eigenvalues of the form (3.5) with multiplicity 4 for the generic case $j > 0$ and 2 for the case $j = 0$. Thus

$$N(t) = \frac{1}{4}\left(N_{T'}(t) - N_T(t)\right) \pm \frac{1}{2}\left[\frac{b}{\pi}t^{1/2} + \frac{1}{2}\right] \tag{3.6}$$

(+ for NM and − for DM cases), since the condition $\frac{\pi^2(2k+1)^2}{4b^2} \leq t$ means $0 \leq k \leq \left[\frac{b}{\pi}t^{1/2} - \frac{1}{2}\right]$. Thus we choose the refined asymptotics

$$\begin{cases} \widetilde{N}_{NM}(t) = \dfrac{ab}{4\pi}t + \dfrac{2b}{4\pi}t^{1/2} \\ \widetilde{N}_{DM}(t) = \dfrac{ab}{4\pi}t - \dfrac{2b}{4\pi}t^{1/2}, \end{cases} \tag{3.7}$$

and the analog of Theorem 2.2 holds. Indeed $\widetilde{N}_{T'}(t) - \widetilde{N}_T(t) = \frac{8ab}{4\pi}t - \frac{4ab}{4\pi}t$, and $\left[\frac{b}{\pi}t^{1/2} + \frac{1}{2}\right] - \frac{b}{\pi}t^{1/2} = O(t^{-1/2})$ as in the proof of Theorem 2.2. The explanation for the coefficient of $t^{1/2}$ in (3.7) is that the sides of length a cancel because they have mixed boundary conditions, while the sides of length b add because they have like boundary conditions. There is no constant term because there are two vertices with mixed boundary conditions and two with like boundary conditions on their incident edges.

The last case of mixed boundary conditions on R has mixed conditions on both pairs of opposite edges (MM). In this case we need a still larger torus T'' of size $4a \times 4b$ on which to extend the eigenfunctions. Then the eigenfunctions have the form

$$f(j,k) = \cos\pi\left(\frac{j+\frac{1}{2}}{a}\right)x \cos\pi\left(\frac{k+\frac{1}{2}}{b}\right)y \qquad \text{for } j \geq 0, k \geq 0 \tag{3.8}$$

with eigenvalue

$$\frac{\pi^2(2j+1)^2}{4a^2} + \frac{\pi^2(2k+1)^2}{4b^2}. \tag{3.9}$$

If we denote by T'_1 and T'_2 the tori of sizes $2a \times 4b$ and $4b \times 2a$, then $N_{T''} - N_{T'_1} - N_{T'_2} + N_T$ counts all eigenvalues of the form (3.9) with multiplicity 4, and all eigenvalues are generic. So

$$N_{MM}(t) = \frac{1}{4}\left(N_{T''}(t) - N_{T'_1}(t) - N_{T'_2}(t) + N_T(t)\right) \tag{3.10}$$

Thus we define the refined asymptotics

$$\widetilde{N}_{MM}(t) = \frac{ab}{4\pi}t, \tag{3.11}$$

and the analog of Theorem 2.2 holds because $\frac{1}{4}\left(\widetilde{N}_{T''}(t) - \widetilde{N}_{T'_1}(t) - \widetilde{N}_{T'_2}(t) + \widetilde{N}_T(t)\right) = \widetilde{N}_{MM}(t)$. Here there is no $t^{1/2}$ term in (3.11) because of cancellation of opposite sides, and no constant term because of cancellation of vertex pairs.

Next we look at flat cylinders C obtained from the rectangles R by identifying the opposite edges of side length b. We impose either Neumann (N), Dirichlet (D) or mixed (M) boundary conditions on the other pair of opposite edges. Then reflection produces eigenfunctions on the torus T_3' of size $a\times 2b$ (N or D) or T_4' of size $a\times 4b$ (M). The eigenfunctions are given by

$$f(j,k) = \begin{cases} e^{2\pi i \frac{j}{a}x}\cos\pi\frac{k}{b}y & \text{for } j\in\mathbb{Z}, k\geq 0 \quad \text{(N)} \\ e^{2\pi i \frac{j}{a}x}\sin\pi\frac{k}{b}y & \text{for } j\in\mathbb{Z}, k> 0 \quad \text{(D)} \\ e^{2\pi i \frac{j}{a}x}\cos\pi\left(\frac{k+\frac{1}{2}}{b}\right)y & \text{for } j\in\mathbb{Z}, k\geq 0 \quad \text{(M)} \end{cases} \tag{3.12}$$

with eigenvalue

$$\begin{cases} \frac{4\pi^2 j^2}{a^2}+\frac{\pi^2 k^2}{b^2} & \text{(N or D)} \\ \frac{4\pi^2 j^2}{a^2}+\frac{\pi^2(2k+1)^2}{4b^2} & \text{(M).} \end{cases} \tag{3.13}$$

Reasoning as before we find

$$\begin{cases} N_N(t) = \frac{1}{2}N_{T_3'}(t)+\left[\frac{at^{1/2}}{2\pi}\right]+\frac{1}{2} \\ N_D(t) = \frac{1}{2}N_{T_3'}(t)-\left[\frac{at^{1/2}}{2\pi}\right]-\frac{1}{2} \\ N_M(t) = \frac{1}{2}\left(N_{T_4'}(t)-N_{T_3'}(t)\right). \end{cases} \tag{3.14}$$

We define the refined asymptotics

$$\begin{cases} \widetilde{N}_N(t) = \frac{ab}{4\pi}t+\frac{2a}{4\pi}t^{1/2} \\ \widetilde{N}_D(t) = \frac{ab}{4\pi}t-\frac{2a}{4\pi}t^{1/2} \\ \widetilde{N}_M(t) = \frac{ab}{4\pi}t, \end{cases} \tag{3.15}$$

and the analog of Theorem 2.2 holds.

Next we consider a Möbius band obtained from the rectangle R by identifying the sides of length a with reversed orientation, with either Neumann or Dirichlet boundary condition on the boundary curve of length $2b$. We can extend eigenfunctions to the torus T, and we find they have the form

$$f(j,k) = \begin{cases} e^{\pi i \frac{j}{a}x}\cos\frac{\pi k}{b}y & \text{if } k\geq 0 \text{ and } j+k \text{ is even} \quad \text{(N)} \\ e^{\pi i \frac{j}{a}x}\sin\frac{\pi k}{b}y & \text{if } k> 0 \text{ and } j+k \text{ is odd} \quad \text{(D)} \end{cases} \tag{3.16}$$

with eigenvalue $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2}$. Consider the lattice counting function

$$\begin{cases} N_e(t) = \#\{(j,k) \,:\, j+k \text{ is even and } \frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2} \le t\} \\ N_o(t) = \#\{(j,k) \,:\, j+k \text{ is odd and } \frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2} \le t\}. \end{cases} \tag{3.17}$$

Reasoning as before we find

$$\begin{cases} N_N(t) = \frac{1}{2} N_e(t) + \left[\frac{at^{1/2}}{2\pi}\right] + \frac{1}{2} \\ N_D(t) = \frac{1}{2} N_o(t) - \left[\frac{at^{1/2}}{2\pi} + \frac{1}{2}\right]. \end{cases} \tag{3.18}$$

Now $N_e(t)$ is equal to the counting function corresponding to a torus T_5' of area $2ab$, and $N_o(t) = N_T(t) - N_{T_5'}(t)$, so both $\frac{1}{2}N_e(t)$ and $\frac{1}{2}N_o(t)$ are covered by the results of [JS]. Thus we define the refined asymptotics

$$\begin{cases} \widetilde{N}_N(t) = \frac{ab}{4\pi} t + \frac{2a}{4\pi} t^{1/2} \\ \widetilde{N}_D(t) = \frac{ab}{4\pi} t - \frac{2a}{4\pi} t^{1/2}, \end{cases} \tag{3.19}$$

with the analog of Theorem 2.2 holding. Note that the refined asymptotics of the Möbius band is identical to that of the cylinder, although the eigenvalue counting functions are not equal, and the almost-periodic function g is not the same.

Next we consider the isosceles triangle with mixed boundary conditions. There are essentially four different cases. In the first two we consider the same type of boundary condition on the equal sides, and the opposite type on the hypotenuse. We write (ND) for Neumann on the equal sides, and (DN) for Dirichlet on the equal sides. In the ND case we take an odd reflection in the hypotenuse to obtain a pure Neumann condition on the square. If we compare with Lemma 2.3 where we took an even reflection, we see that the eigenfunctions on the square with Neumann conditions split into the mixed ND eigenfunctions and the pure N eigenfunctions, so

$$N_{ND}(t) + N_N(t) = N_N^{(S)}(t), \tag{3.20}$$

where the right side denotes the counting function for the square. Similarly

$$N_{DN}(t) + N_D(t) = N_D^{(S)}(t). \tag{3.21}$$

Thus, $N_{ND}(t)$ is given by the difference of (2.6) (with $b = a$) and (2.15), so

$$N_{ND}(t) = \frac{1}{8}N_T(t) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{at^{1/2}}{\sqrt{2}\pi}\right] + \frac{1}{2}\right) - \frac{1}{8}. \quad (3.22)$$

Similarly N_{DN} is the difference of (2.7) and (2.15), so

$$N_{DN}(t) = \frac{1}{8}N_T(t) - \frac{1}{2}\left(\left[\frac{at^{1/2}}{\pi}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{at^{1/2}}{\sqrt{2}\pi}\right] + \frac{1}{2}\right) - \frac{1}{8}. \quad (3.23)$$

Here the constant is $\psi(\frac{\pi}{2}) + 2\left(\psi(\frac{\pi}{2}) - \psi(\frac{\pi}{4})\right) = \frac{3}{16} - 2(\frac{5}{32}) = -\frac{1}{8}$.

This leads to the choice

$$\widetilde{N}_{ND}(t) = \frac{a^2}{8\pi}t + \left(\frac{2-\sqrt{2}}{4\pi}\right)at^{1/2} - \frac{1}{8}, \quad (3.24)$$

$$\widetilde{N}_{DN}(t) = \frac{a^2}{8\pi}t + \left(\frac{-2+\sqrt{2}}{4\pi}\right)at^{1/2} - \frac{1}{8}, \quad (3.25)$$

and the analog of Theorem 2.4 holds.

In the last two cases we have mixed boundary conditions on the two equal sides and Neumann (MN) or Dirichlet (MD) conditions on the hypotenuse. By reflecting evenly or oddly in the hypotenuse we end up in the (MM) case for the square. As in the earlier cases the (MM) eqigenfunctions on the square (3.8) yield the (MN) eigenfunctions $f(j,k) + f(k,j)$ and the (MD) eigenfunctions $f(j,k) - f(k,j)$ $(k \neq j)$. In the place of (3.20) and (3.21) we have

$$N_{MN}(t) + N_{MD}(t) = N_{MM}^{(S)}(t), \quad (3.26)$$

but this does not allow us to immediately compute the two summands. However, in the generic case $j \neq k$ we obtain one eigenfunction in each case for two in the square, while in the nongeneric case $j = k$ we obtain one in the (MN) case and none in the (MD) case for one in the square. Thus

$$N_{MN}(t) = \frac{1}{2}N_{MM}^{(S)}(t) + \frac{1}{2}\left[\frac{\sqrt{2}a}{2\pi}t^{1/2} + \frac{1}{2}\right] \quad (3.27)$$

and

$$N_{MD}(t) = \frac{1}{2} N_{MM}^{(S)}(t) - \frac{1}{2}\left[\frac{\sqrt{2}a}{2\pi} t^{1/2} + \frac{1}{2}\right]. \tag{3.28}$$

This leads to the choice

$$\widetilde{N}_{MN}(t) = \frac{a^2}{8\pi} t + \left(\frac{\sqrt{2}}{4\pi}\right) at, \tag{3.29}$$

$$\widetilde{N}_{MD}(t) = \frac{a^2}{8\pi} t - \left(\frac{\sqrt{2}}{4\pi}\right) at, \tag{3.30}$$

with the analog of Theorem 2.4 holding. Here the constant is $\big(\psi(\pi) - \psi(\frac{\pi}{2})\big) + \psi(\frac{\pi}{4}) + \big(\psi(\frac{\pi}{2}) - \psi(\frac{\pi}{4})\big) = 0$.

In our last example we consider two mixed boundary problems on the $30° - 60° - 90°$ triangle with the same type of boundary condition on the hypotenuse and the shortest side, and the opposite type on the side that cuts the equilateral triangle in half that we call (ND) and (DN). In the (ND) case an odd reflection across the cut side produces a Neumann boundary condition on the equilateral triangle, so these eigenfunctions together with the pure Neumann eigenfunctions split up the pure Neumann eigenfunctions on the equilateral triangle. Thus $N_{ND}(t)$ is given by the difference of (2.22) and (2.26), hence

$$\begin{aligned} N_{ND}(t) = \frac{1}{6} N_T(t) + \left(\left[\frac{3}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) + \frac{1}{3} \\ -\frac{1}{12}\left(N_T(t) + \frac{1}{2}\left(\left[\frac{3}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{\sqrt{3}}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) + \frac{5}{12}\right) \\ = \frac{1}{12} N_T(t) + \frac{1}{2}\left(\left[\frac{3}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{\sqrt{3}}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) - \frac{1}{12}. \end{aligned} \tag{3.31}$$

Similarly $N_{DN}(t)$ is the difference of (2.23) and (2.27), hence

$$N_{DN}(t) = \frac{1}{12} N_T(t) - \frac{1}{2}\left(\left[\frac{3}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{\sqrt{3}}{4\pi} t^{1/2}\right] + \frac{1}{2}\right) - \frac{1}{12}. \tag{3.32}$$

This leads to the choice

$$\widetilde{N}_{ND}(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{8}t + \frac{3-\sqrt{3}}{8\pi}t^{1/2} - \frac{1}{12}, \tag{3.33}$$

$$\widetilde{N}_{DN}(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{8}t + \frac{-3+\sqrt{3}}{8\pi}t^{1/2} - \frac{1}{12}, \tag{3.34}$$

with the analog of Theorem 2.2 holding. Here the constant is $\big(\psi(\pi)-\psi(\frac{\pi}{2})\big)+\psi(\frac{\pi}{3})+\big(\psi(\frac{\pi}{3})-\psi(\frac{\pi}{6})\big) = -\frac{1}{16}+\frac{2}{9}-\frac{35}{144} = -\frac{1}{12}$.

It does not appear that mixed boundary problems on the equilateral triangle or the other types of mixed boundary problems on the $30° - 60° - 90°$ triangle have eigenfunctions that may all be described by trigonometric polynomials.

4 The Sphere

We consider the 2-dimensional sphere S^2, and for simplicity we take the radius equal to 1. The eigenvalues of the Laplacian are given by the theory of spherical harmonics, with eigenvalue $k(k+1)$ having multiplicity $2k+1$, for $k = 0, 1, \ldots$ The eigenfunctions are the restriction to S^2 of homogeneous harmonic polynomials of degree k. We thus have

$$N(t) = k^2 \quad \text{for} \quad k^2 - k \le t < k^2 + k, \tag{4.1}$$

which we simplify to

$$N(t) = \left[\sqrt{t+\frac{1}{4}} + \frac{1}{2}\right]^2. \tag{4.2}$$

Now we define the refined asymptotics

$$\widetilde{N}(t) = t + \frac{1}{3}. \tag{4.3}$$

Lemma 4.1. *For $k^2 - k \le t < k^2 + k$ we have*

$$A(t) = \frac{1}{2t}\left(k^2 - (k^2-t)^2\right) - \frac{1}{3} \tag{4.4}$$

Proof. $A(t) = \dfrac{1}{t}\displaystyle\int_0^t (N(s)-s)\,ds - \dfrac{1}{3}$. Now for $j < k$ we observe that $\displaystyle\int_{j^2-j}^{j^2+j} (N(s)-s)\,ds = \int_{j^2-j}^{j^2+j} (j^2-s)\,ds = 0$ because $j^2 - s$ is a linear function on the interval varying between j and $-j$. Thus $A(t) = \dfrac{1}{t}\displaystyle\int_{k^2-k}^{t} (k^2-s)\,ds - \dfrac{1}{3}$, and when we evaluate the integral we obtain (4.4). □

Theorem 4.2. *Let*

$$g(x) = \frac{1}{6} - 2(x - [x + \frac{1}{2}])^2. \tag{4.5}$$

Then g is a continuous periodic function of period 1 with mean value zero, and

$$A(t) = g(\sqrt{t + \frac{1}{4}}) + O(t^{-1/2}). \tag{4.6}$$

Proof. It is clear from the definition that g is continuous and periodic, and if $x = k + r$ with $-\frac{1}{2} \le r < \frac{1}{2}$ then $g(x) = \frac{1}{6} - 2r^2$. One easily checks that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{6} - 2r^2)\,dr = 0$, so g has a mean value zero. □

If we write $\sqrt{t + \frac{1}{4}} = k + r$ then $k^2 = t + \frac{1}{4} - 2r\sqrt{t + \frac{1}{4}} + r^2$ and $(k^2 - t)^2 = \left(\frac{1}{4} + r^2 - 2r\sqrt{t + \frac{1}{4}}\right)^2 = (\frac{1}{4} + r^2)^2 - r(1 + 4r^2)\sqrt{t + \frac{1}{4}} + 4r^2(t + \frac{1}{4})$ so $k^2 - (k^2 - t)^2 = t(1 - 4r^2) - \sqrt{t + \frac{1}{4}}(1 - 4r^2) + \frac{1}{4} + r^2 - (\frac{1}{4} + r^2)^2 - r^2$ $= t(1 - 4r^2) - \sqrt{t + \frac{1}{4}}r(1 - 4r^2) + \frac{(4r^2+3)(1-4r^2)}{16}$. Thus (4.4) becomes

$$A(t) = \frac{1}{6} - 2r^2 - \frac{r(1 - 4r^2)}{2}\frac{\sqrt{t + \frac{1}{4}}}{t} + \frac{(4r^2 + 3)(1 - 4r^2)}{32t}. \tag{4.7}$$

Corollary 4.3. *There exist continuous periodic functions g_1 and g_2 such that*

$$A(t) = g\left(\sqrt{t + \frac{1}{4}}\right) + g_1\left(\sqrt{t + \frac{1}{4}}\right)\frac{\sqrt{t + \frac{1}{4}}}{t} + \frac{g_2\left(\sqrt{t + \frac{1}{4}}\right)}{t}. \tag{4.8}$$

More precisely,

$$g_1(x) = \frac{\left(x - \left[x + \frac{1}{2}\right]\right)\left(4\left(x - \left[x - \frac{1}{2}\right]\right)^2\right)}{2} \quad \textit{and} \tag{4.9}$$

$$g_2(x) = \frac{1}{32} 4\left(x - \left[x + \frac{1}{2}\right]\right)^2 + 3)\left(1 - 4\left(x - \left[x + \frac{1}{2}\right]\right)^2\right). \tag{4.10}$$

Proof. When $x = \sqrt{t + \frac{1}{4}}$ we have $r = x - \left[x + \frac{1}{2}\right]$, so (4.8) is exactly (4.7). $\square$

Next we consider the hemisphere H, with either Neumann or Dirichlet boundary conditions. Reflecting an eigenfunction symmetrically or skew-symmetrically about the boundary produces an eigenfunction on S^2, so the eigenvalues are the same $k(k+1)$, and the multiplicity $2k+1$ on S^2 splits into $k+1$ for Neumann and k for Dirichlet boundary conditions on H. The analog of (4.1) is thus

$$N_N(t) = \frac{k^2 + k}{2}, \quad N_D(t) = \frac{k^2 - k}{2}. \tag{4.11}$$

For this reason we choose

$$\begin{cases} \widetilde{N}_N(t) = \frac{1}{2}t + \frac{1}{2}\sqrt{t + \frac{1}{4}} + \frac{1}{6} \\ \widetilde{N}_D(t) = \frac{1}{2}t - \frac{1}{2}\sqrt{t + \frac{1}{4}} + \frac{1}{6}. \end{cases} \tag{4.12}$$

We note that we could replace $\sqrt{t + \frac{1}{4}}$ by $\sqrt{t}$, as that would only change the refined asymptotics by $O(t^{-1/2})$ and this would get absorbed into the remainder term in $A(t)$.

Theorem 4.4. *Let $g(t)$ be as in Theorem 4.2. Then*

$$A(t) = \frac{1}{2} g\left(\sqrt{t + \frac{1}{4}}\right) + O(t^{-1/2}) \tag{4.13}$$

for either boundary condition.

Proof. We may write $A(t) = \frac{1}{2}A_1(t) \pm \frac{1}{2}A_2(t)$ where A_1 is the average function for the sphere, and

$$A_2(t) = \frac{1}{t}\int_0^t \left(j - \sqrt{s + \frac{1}{4}}\right) ds$$

where $\sqrt{s+\frac{1}{4}} = j + r$ with $-\frac{1}{2} \le r < \frac{1}{2}$. We break the integral up into the intervals $j^2 - j \le s < j^2 + j$ for $j < k$, and the final interval $k^2 - k \le s \le t$. Now $\displaystyle\int_{j^2-j}^{j^2+j} \left(j - \sqrt{s + \frac{1}{4}} \right) ds = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2r(j+r)\, dr = \frac{1}{6}$, so the contributions from the intervals $j < k$ add up to $\frac{1}{6t}(k-1) = O(t^{-1/2})$. For the final interval we note that the integrand is bounded and the interval has length at most $2k$, so its contribution is also $O(\frac{k}{t}) = O(t^{-1/2})$. Thus $A_2(t) = O(t^{-1/2})$. $\square$

We note that the $\sqrt{t+\frac{1}{4}}$ coefficient in (4.12) is exactly $\frac{2}{4\pi}$ where $L = 2\pi$ is the length of the boundary circle of H. The circle does not contribute to the constant terms because it is a geodesic in the sphere.

Finally we consider the projective sphere PS^2 obtained by identifying antipodal points in S^2. Then the eigenfunctions are exactly the spherical harmonics of even degree. Thus for $k^2 - k \le t < k^2 - k$ we have

$$N(t) = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} (4j+1) = \begin{cases} \frac{k^2+k}{2} & \text{if } k \text{ is even.} \\ \frac{k^2-k}{2} & \text{if } k \text{ is odd.} \end{cases} \tag{4.14}$$

Thus we define the refined asymptotics

$$\widetilde{N}(t) = \frac{1}{2}t + \frac{1}{6}, \tag{4.15}$$

and the analog of Theorem 4.4 holds, since the average value of the function $(-1)^k k$ is $O(t^{-1/2})$.

5 Lunes

Fix an even number $2m$, and consider the lune L_{2m} cut from the sphere by two great circles from pole to pole separated by the angle $\frac{2\pi}{2m}$. Let (x, y, w) denote the coordinates in $\mathbb{R}^3$. We may also think of spherical harmonics as polynomials in $(z, \bar{z}, w)$. Eigenfunctions on the lune with Neumann boundary conditions extend by even reflection to the whole sphere, and so are given by spherical harmonics that are invariant under the rotation $z \mapsto e^{\frac{2\pi i}{m}} z$ and

the reflection $z \mapsto \bar{z}$. Similarly, with Dirichlet boundary conditions and odd reflection the spherical harmonics muct be invariant under the rotation and skew-symmetric under the reflection.

Fix a nonnegative integer N, and let $\mathbb{P}_N$ denote the space of polynomial (not necessarily harmonic) that are homogeneous of degree N and satisfy the conditions

$$f(e^{\frac{2\pi i}{m}} z, e^{-\frac{2\pi i}{m}} \bar{z}, w) = f(z, \bar{z}, w) \qquad \text{and} \tag{5.1}$$

$$f(\bar{z}, z, w) = f(z, \bar{z}, w). \tag{5.2}$$

It is clear that the functions $|z|^{2j}(z^m + \bar{z}^m)^k w^\ell$, for $2j + mk + \ell = N$, belong to $\mathbb{P}_N$, and in fact give a basis for $\mathbb{P}_N$. Similarly if $\mathbb{Q}_N$ denotes the space of polynomials homogeneous of degree N satisfying (5.1) and

$$f(\bar{z}, z, w) = -f(z, \bar{z}, w) \tag{5.3}$$

instead of (5.2) then $\mathbb{Q}_N$ consists of functions of the form $(z^m - \bar{z}^m) f_1(z, \bar{z}, w)$ for $f_1 \in \mathbb{P}_{N-m}$ (in particular $\mathbb{Q}_N$ is $\{0\}$ unless $N \geq m$). We also observe that $\Delta : \mathbb{P}_N \to \mathbb{P}_{N-2}$ and $\Delta : \mathbb{Q}_N \to \mathbb{Q}_{N-2}$ and both mappings are onto. It follows that the multiplicities of the eigenvalue $N(N+1)$ in the Neumann and Dirichlet spectra of L_{2m} are given by $\dim \mathbb{P}_N - \dim \mathbb{P}_{N-2}$ and $\dim \mathbb{Q}_N - \dim \mathbb{Q}_{N-2}$ respectively. It turns out to be easier to compute these differences of dimensions rather than $\dim \mathbb{P}_N$ and $\dim \mathbb{Q}_N$.

Lemma 5.1. *For any $N \geq 0$ we have*

$$\dim \mathbb{P}_N - \dim \mathbb{P}_{N-2} = \left[\frac{N}{m}\right] + 1, \tag{5.4}$$

while for any $N \geq m$ we have

$$\dim \mathbb{Q}_N - \dim \mathbb{Q}_{N-2} = \left[\frac{N}{m}\right]. \tag{5.5}$$

Proof. The mapping $f \mapsto |z|^2 f$ maps $\mathbb{P}_{N-2}$ into $\mathbb{P}_N$ and is one-to-one, so $\dim \mathbb{P}_N - \dim \mathbb{P}_{N-2}$ is equal to the dimension of the complementary space, which is spanned by $(z^m + \bar{z}^m)^k w^\ell$ for $mk + \ell = N$. For each choice of k with $mk \leq N$ there is a unique choice of ℓ, namely $\ell = N - mk$. But the number of choices of k with $mk \leq N$ is exactly $\left[\frac{N}{m}\right] + 1$, proving (5.4). Then $\dim \mathbb{Q}_N - \dim \mathbb{Q}_{N-2} = \dim \mathbb{P}_{N-m} - \dim \mathbb{P}_{N-m-1} = \left[\frac{N-m}{m}\right] + 1 = \left[\frac{N}{m}\right]$. $\square$

Let $N_N(t)$ and $N_D(t)$ denote the eigenvalue counting functions for the Neumann and Dirichlet eigenfunctions on L_{2m}. Then for $k^2 - k \le t < k^2 + k$ we have by Lemma 5.1 that

$$N_N(t) = \sum_{j=0}^{k-1} \left(\left[\frac{j}{m} \right] + 1 \right) \qquad \text{and} \tag{5.6}$$

$$N_D(t) = \sum_{j=m}^{k-1} \left[\frac{j}{m} \right] \tag{5.7}$$

Lemma 5.2. *Suppose $k \equiv p \mod m$ and $k^2 - k \le t < k^2 + k$. Then*

$$N_N(t) = \frac{k^2}{2m} + \frac{k}{2} + \frac{p(m-p)}{2m} \qquad \textit{and} \tag{5.8}$$

$$N_D(t) = \frac{k^2}{2m} - \frac{k}{2} + \frac{p(m-p)}{2m}. \tag{5.9}$$

Proof. The sum in (5.6) repeats each integer from 1 up to $\left[\frac{k}{m}\right]$ exactly m times, and then $\left[\frac{k}{m}\right]+1$ exactly p times. Note that $k = p + m\left[\frac{k}{m}\right]$, so $N_N(t) = \frac{m}{2}\left[\frac{k}{m}\right]\left(\left[\frac{k}{m}\right]+1\right) + p\left(\left[\frac{k}{m}\right]+1\right) = \left(\left[\frac{k}{m}\right]+1\right)\left(\frac{m}{2}\left[\frac{k}{m}\right]+p\right) = \left(\frac{k-p+m}{m}\right)\left(\frac{k+p}{2}\right)$ which yields (5.8). Then (5.9) follows since $N_D(t) = N_N(t) - k$ by (5.6) and (5.7). $\square$

Now we note that the average value of the constant term $\frac{p(m-p)}{2m}$ as p varies from 0 to $m-1$ is

$$\begin{aligned} \frac{1}{2m^2}\sum_{p=0}^{m-1}(mp - p^2) &= \frac{m^2(m-1)}{4m^2} - \frac{m(m-1)(2m-1)}{12m^2} \\ &= \frac{1}{12}\left(m - \frac{1}{m}\right). \end{aligned}$$

This leads to the choice of refined asymptotics

$$\widetilde{N}(t) = \frac{t}{2m} \pm \frac{\sqrt{t+\frac{1}{4}}}{2} + \frac{1}{12}\left(m - \frac{1}{m}\right) + \frac{1}{6m} \tag{5.10}$$

(+ for $\widetilde{N}_N$ and − for $\widetilde{N}_D$). Note that we could also replace $\sqrt{t+\frac{1}{4}}$ by $\sqrt{t}$ since the difference is $O(t^{-1/2})$.

Theorem 5.3. *There exists a continuous, periodic function g of mean value zero such that*

$$A(t) = g(\sqrt{t + \frac{1}{4}}) + O(t^{-1/2}). \tag{5.11}$$

Proof. We may write $D(t) = N(t) - \widetilde{N}(t)$ as $D_1(t) + D_2(t) + D_3(t)$ where

$$\begin{cases} D_1(t) &= \frac{k^2}{2m} - \left(\frac{t}{2m} + \frac{1}{6m}\right), \\ D_2(t) &= \pm\left(\frac{k}{2} - \frac{\sqrt{t+\frac{1}{4}}}{2}\right), \qquad \text{and} \\ D_3(t) &= \frac{p(m-p)}{2m} - \frac{1}{12}\left(m - \frac{1}{m}\right). \end{cases} \tag{5.12}$$

If we form the corresponding average functions A_1, A_2, A_3 it suffices to show (5.11) for each separately. Now the result for A_1 is Theorem 4.2 multiplied by $\frac{1}{2m}$. For A_2 it is easy to see that in fact $A_2(t) = O(t^{-1/2})$. It remains to consider A_3.

Assume $k^2 - k \le t < k^2 + k$ for $k = jm + p$ with $1 \le p \le m$. We may write

$$A_3(t) = \frac{1}{t}\sum_{n=0}^{j-1}\sum_{q=1}^{m}\int_{(nm+q)^2-(nm+q)}^{(nm+q)^2+(nm+q)} D_3(t)\,dt + \frac{1}{t}\int_{(jm+1)^2-(jm+1)}^{t} D_3(t)\,dt.$$

It is easy to see that the last term is $O(t^{-1/2})$ because the integrand is bounded and the length of the interval is $O(t^{1/2})$. In each integral in the sum $D_3(t)$ is constant, so

$$\begin{aligned} A_3(t) &= \frac{1}{t}\sum_{n=0}^{j-1}\sum_{q=1}^{m} 2(nm+q)\left(\frac{q(m-q)}{2m} - \frac{1}{12}\left(m - \frac{1}{m}\right)\right) + O(t^{-1/2}) \\ &= \frac{1}{t}\sum_{n=0}^{j-1}\sum_{q=1}^{m} 2q\left(\frac{q(m-q)}{2m} - \frac{1}{12}\left(m - \frac{1}{m}\right)\right) + O(t^{-1/2}) \end{aligned}$$

since $\sum_{q=1}^{m} 2nm\left(\frac{q(m-q)}{2m} - \frac{1}{12}\left(m - \frac{1}{m}\right)\right) = 0$. But the summands are uniformly bounded and the number of terms is $O(t^{1/2})$, so $A_3(t) = O(t^{-1/2})$. $\square$

We observe (5.10) has the predicted form. Conpared with (4.3) for the sphere, the leading term is reduced by the factor $\frac{1}{2m}$ because the area of the lune is $\frac{1}{2m}$ times the area of the circle, and the $\pm\frac{1}{2}$ factor in the square root term is $\frac{2\pi}{4\pi}$, where 2π is the length of the perimeter of the lune. Finally, the constant is twice $\frac{1}{24}\left(m-\frac{1}{m}\right)$, the value predicted for each of the two angels $\frac{\pi}{m}$ of the lune. We could also obtain the analog of corollary 4.3 for the remainder term in (5.11).

Next we consider the half-lune $L_{2m}^{1/2}$ obtained by slicing along the equator. Functions satisfying Neumann or Dirichlet conditions on the equatorial edge extend to the entire lune by even or odd reflection; so we may study four different types of boundary conditions. We denote by N_N^+, N_N^-, N_D^+ and N_D^- the corresponding eigenvalue counting functions, where the $\pm$ denote the equatorial boundary conditions. Then $N_N(t)=N_N^+(t)+N_N^-(t)$ and $N_D(t)=N_D^+(t)+N_D^-(t)$, and we need to understand how the spherical harmonics that contribute to $N_N(t)$ and $N_D(t)$ split into even and odd functions in the w variable. It is clear that this is determined by the parity of ℓ in the basis element $|z|^{2j}(z^m+\bar{z}^m)^k w^\ell$.

Lemma 5.4. *(a) Assume m is even. Then*

$$\dim \mathbb{P}_N^+ - \dim \mathbb{P}_{N-2}^+ = \begin{cases} \left[\frac{N}{m}\right]+1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd} \end{cases} \tag{5.13}$$

$$\dim \mathbb{P}_N^- - \dim \mathbb{P}_{N-2}^- = \begin{cases} 0 & \text{if } N \text{ is even} \\ \left[\frac{N}{m}\right]+1 & \text{if } N \text{ is odd} \end{cases} \tag{5.14}$$

$$\dim \mathbb{Q}_N^+ - \dim \mathbb{Q}_{N-2}^+ = \begin{cases} \left[\frac{N}{m}\right] & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd} \end{cases} \tag{5.15}$$

$$\dim \mathbb{Q}_N^- - \dim \mathbb{Q}_{N-2}^- = \begin{cases} 0 & \text{if } N \text{ is even} \\ \left[\frac{N}{m}\right] & \text{if } N \text{ is odd} \end{cases} \tag{5.16}$$

(b) Assume m is odd. Then

$$\dim \mathbb{P}_N^+ - \dim \mathbb{P}_{N-2}^+ = \begin{cases} \frac{1}{2}\left[\frac{N}{m}\right]+\frac{1}{2} & \text{if } \left[\frac{N}{m}\right] \text{ is odd} \\ \frac{1}{2}\left[\frac{N}{m}\right]+1 & \text{if } \left[\frac{N}{m}\right] \text{ is even and } N \text{ is even} \\ \frac{1}{2}\left[\frac{N}{m}\right] & \text{if } \left[\frac{N}{m}\right] \text{ is even and } N \text{ is odd} \end{cases} \tag{5.17}$$

$$\dim \mathbb{P}_N^- - \dim \mathbb{P}_{N-2}^- = \begin{cases} \frac{1}{2}\left[\frac{N}{m}\right] + \frac{1}{2} & \textit{if } \left[\frac{N}{m}\right] \textit{ is odd} \\ \frac{1}{2}\left[\frac{N}{m}\right] & \textit{if } \left[\frac{N}{m}\right] \textit{ is even and } N \textit{ is even} \\ \frac{1}{2}\left[\frac{N}{m}\right] + 1 & \textit{if } \left[\frac{N}{m}\right] \textit{ is even and } N \textit{ is odd} \end{cases} \tag{5.18}$$

$$\dim \mathbb{Q}_N^+ - \dim \mathbb{Q}_{N-2}^+ = \begin{cases} \frac{1}{2}\left[\frac{N}{m}\right] & \textit{if } \left[\frac{N}{m}\right] \textit{ is even} \\ \frac{1}{2}\left[\frac{N}{m}\right] + \frac{1}{2} & \textit{if } \left[\frac{N}{m}\right] \textit{ is odd and } N \textit{ is odd} \\ \frac{1}{2}\left[\frac{N}{m}\right] - \frac{1}{2} & \textit{if } \left[\frac{N}{m}\right] \textit{ is odd and } N \textit{ is even} \end{cases} \tag{5.19}$$

$$\dim \mathbb{Q}_N^- - \dim \mathbb{Q}_{N-2}^- = \begin{cases} \frac{1}{2}\left[\frac{N}{m}\right] & \textit{if } \left[\frac{N}{m}\right] \textit{ is even} \\ \frac{1}{2}\left[\frac{N}{m}\right] - \frac{1}{2} & \textit{if } \left[\frac{N}{m}\right] \textit{ is odd and } N \textit{ is odd} \\ \frac{1}{2}\left[\frac{N}{m}\right] + \frac{1}{2} & \textit{if } \left[\frac{N}{m}\right] \textit{ is odd and } N \textit{ is even} \end{cases} \tag{5.20}$$

Proof. As in the proof of Lemma 5.1, we need to count the number of solutions of $mk + \ell = N$ but now with ℓ restricted to be even and odd. If m is even then $\ell = N - mk$ will have the same parity as N, and all $\left[\frac{N}{m}\right] + 1$ choices of k are allowed. This proves (5.13) and (5.14). A similar argument proves (5.15) and (5.16) since there are only $\left[\frac{N}{m}\right]$ choices of k. If m is odd, then ℓ will be even if N and k have the same parity, and ℓ will be odd if N and k have opposite parity. If N is even, then the even choices of k in $0 \leq k \leq \left[\frac{N}{m}\right]$ will contribute to $\dim \mathbb{P}_N^+ - \dim \mathbb{P}_{N-2}^+$ while the odd choices of k will contribute to $\dim \mathbb{P}_N^- - \dim \mathbb{P}_{N-2}^-$. If $\left[\frac{N}{m}\right]$ is odd there will be exactly $\frac{1}{2}\left[\frac{N}{m}\right] + \frac{1}{2}$ of even and odd choices, while if $\left[\frac{N}{m}\right]$ is even there will be $\frac{1}{2}\left[\frac{N}{m}\right] + 1$ even choices and $\frac{1}{2}\left[\frac{N}{m}\right]$ odd choices. If, on the other hand, N is odd, it is the odd choices of k that contribute to $\dim \mathbb{P}_N^+ - \dim \mathbb{P}_{N-2}^+$ and the even choices of k that contribute to $\dim \mathbb{P}_N^- - \dim \mathbb{P}_{N-2}^-$. Putting this together yields (5.17) and (5.18). Then (5.19) and (5.20) are obtained by replacing N by $N - m$, which flips the parity of both N and $\left[\frac{N}{m}\right]$. □

Lemma 5.5. *Suppose $k \equiv p \mod m$ and $k^2 - k \leq t < k^2 + k$.*
(a) Assume m is even. Then

$$N_N^+(t) = \begin{cases} \frac{k^2}{4m} + \frac{k}{4} + \frac{(m-p)p}{4m} & \textit{if } p \textit{ is even} \\ \frac{k^2}{4m} + \left(\frac{1}{4} + \frac{1}{2m}\right)k + \frac{(m-p)p}{4m} + \frac{m-p}{2m} & \textit{if } p \textit{ is odd} \end{cases} \tag{5.21}$$

$$N_N^-(t) = \begin{cases} \frac{k^2}{4m} + \frac{k}{4} + \frac{(m-p)p}{4m} & \textit{if } p \textit{ is even} \\ \frac{k^2}{4m} + \left(\frac{1}{4} - \frac{1}{2m}\right)k + \frac{(m-p)p}{4m} + \frac{m-p}{2m} & \textit{if } p \textit{ is odd} \end{cases} \tag{5.22}$$

$$N_D^+(t) = \begin{cases} \frac{k^2}{4m} - \frac{k}{4} + \frac{(m-p)p}{4m} & \textit{if } p \textit{ is even} \\ \frac{k^2}{4m} - \left(\frac{1}{4} - \frac{1}{2m}\right)k + \frac{(m-p)p}{4m} - \frac{p}{2m} & \textit{if } p \textit{ is odd} \end{cases} \tag{5.23}$$

$$N_D^-(t) = \begin{cases} \frac{k^2}{4m} - \frac{k}{4} + \frac{(m-p)p}{4m} & \textit{if } p \textit{ is even} \\ \frac{k^2}{4m} - \left(\frac{1}{4} + \frac{1}{2m}\right)k + \frac{(m-p)p}{4m} + \frac{p}{2m} & \textit{if } p \textit{ is odd} \end{cases} \tag{5.24}$$

(b) Assume m is odd. Then

$$N_N^+(t) = \frac{k^2}{4m} + \left(\frac{1}{4} + \frac{1}{4m}\right)k + \frac{p(m-p)}{4m} + \frac{m-p}{4m} + h(n,p) \tag{5.25}$$

$$N_N^-(t) = \frac{k^2}{4m} + \left(\frac{1}{4} - \frac{1}{4m}\right)k + \frac{p(m-p)}{4m} - \frac{m-p}{4m} - h(n,p) \tag{5.26}$$

$$N_D^+(t) = \frac{k^2}{4m} + \left(-\frac{1}{4} + \frac{1}{4m}\right)k + \frac{p(m-p)}{4m} + \frac{m-p}{4m} + \frac{1}{2} + h(n,p) \tag{5.27}$$

$$N_D^-(t) = \frac{k^2}{4m} + \left(-\frac{1}{4} - \frac{1}{4m}\right)k + \frac{p(m-p)}{4m} - \frac{m-p}{4m} + \frac{1}{2} - h(n,p) \tag{5.28}$$

where

$$h(n,p) = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ \frac{1}{4} & \textit{if } n \textit{ is even and } p \textit{ is odd} \\ -\frac{1}{4} & \textit{if } n \textit{ is even and } p \textit{ is even.} \end{cases} \tag{5.29}$$

Proof. As in the proof of Lemma 5.2, we need to sum the expressions in Lemma 5.4 for $N \le k-1$. Assume m is even. Write $k = mn + p$. Then

$$N_N^+(t) = \sum_{j \le \frac{k-1}{2}} \left(\left[\frac{2j}{m}\right] + 1 \right) \quad \text{and}$$

$$N_N^-(t) = \sum_{j \le \frac{k-2}{2}} \left(\left[\frac{2j+1}{m}\right] + 1 \right).$$

The sum for N_N^+ has all integers from 1 to n repeated $\frac{m}{2}$ times, and the integer $n+1$ repeated $\frac{p}{2}$ times if p is even and $\frac{p+1}{2}$ times if p is odd. Thus

$$N_N^+(t) = \begin{cases} \frac{m}{4}n(n+1) + \frac{p}{2}(n+1) & \text{if } p \text{ is even} \\ \frac{m}{4}n(n+1) + \frac{p+1}{2}(n+1) & \text{if } p \text{ is odd.} \end{cases}$$

Substituting $n = \frac{k-p}{m}$ we obtain (5.21). Similar reasoning shows

$$N_N^-(t) = \begin{cases} \frac{m}{4}n(n+1) + \frac{p}{2}(n+1) & \text{if } p \text{ is even} \\ \frac{m}{4}n(n+1) + \frac{p-1}{2}(n+1) & \text{if } p \text{ is odd.} \end{cases}$$

and this reduces to (5.22).

Similarly we have

$$N_D^+(t) = \sum_{j \le \frac{k-1}{2}} \left[\frac{2j}{m}\right] \quad \text{and}$$

$$N_D^-(t) = \sum_{j \le \frac{k-2}{2}} \left[\frac{2j+1}{m}\right]$$

so we just have to replace n by $n-1$ in the expressions for $N_N^+(t)$ and $N_N^-(t)$. This yields (5.23) and (5.24).

Next consider the case when m is odd. For $N_N^+(t)$ we need to sum over N in $0 \le N \le k-1$ the values $\frac{1}{2}\left[\frac{N}{m}\right] + \frac{1}{2}$, $\frac{1}{2}\left[\frac{N}{m}\right] + 1$ or $\frac{1}{2}\left[\frac{N}{m}\right]$ according to the parity cases in (5.17). As before there will be m choices of N with $\left[\frac{N}{m}\right] = j$ for $0 \le j \le n-1$, and p choices of N with $\left[\frac{N}{m}\right] = n$. So

$$\begin{aligned} N_N^+(t) = m\sum_{j=0}^{n-1} \frac{1}{2}j + \frac{1}{2}pn + \frac{1}{2}\#\{N : \left[\frac{N}{m}\right] \text{ is odd}\} \\ + \#\{N : \left[\frac{N}{m}\right] \text{ is even and } N \text{ is even}\} \end{aligned}$$

If n is odd then

$$\#\{N : \left[\frac{N}{m}\right] \text{ is odd}\} = m\left(\frac{n-1}{2}\right) + p$$

and

$$\#\{N : \left[\frac{N}{m}\right] \text{ is even and } N \text{ is even}\} = \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right).$$

In this case

$$\begin{aligned} N_N^+(t) &= \frac{m(n-1)n}{4} + \frac{pn}{2} + \frac{m(n-1)+(m+1)(n+1)}{4} + \frac{p}{2} \\ &= \frac{m(n+1)n}{4} + \frac{n+1}{4} + \left(\frac{n+1}{2}\right)p, \end{aligned}$$

and substituting $n = \frac{k-p}{m}$ we obtain

$$N_N^+(t) = \frac{k^2}{4m} + \left(\frac{m+1}{4m}\right)k + \frac{p(m-p)}{4m} + \frac{m-p}{4m}.$$

If n is even then

$$\#\{N : \left[\frac{N}{m}\right] \text{ is odd}\} = \frac{mn}{2}, \quad \text{and}$$

$$\#\{N : \left[\frac{N}{m}\right] \text{ is even and } N \text{ is even}\} = \begin{cases} \left(\frac{m+1}{2}\right)\frac{n}{2} + \frac{p+1}{2} & \text{if } p \text{ is odd} \\ \left(\frac{m+1}{2}\right)\frac{n}{2} + \frac{p}{2} & \text{if } p \text{ is even.} \end{cases}$$

So

$$\begin{aligned} N_N^+(t) &= \frac{m(n-1)n}{4} + \frac{pn}{2} + \frac{mn + (m+1)n}{4} + \begin{cases} \frac{p+1}{2} & p \text{ odd} \\ \frac{p}{2} & p \text{ even} \end{cases} \\ &= \frac{m(n+1)n}{4} + \frac{n}{4} + \left(\frac{n+1}{2}\right)p + \frac{1}{2}\,(\text{if } p \text{ is odd}). \end{aligned}$$

This establishes (5.25), and (5.26) follows from $N_N(t) = N_N^+(t) + N_N^-(t)$ and (5.8).

To compute $N_D^\pm(t)$ we note that (5.19) and (5.20) differ from (5.17) and (5.18) by $-\frac{1}{2}$. Thus $N_D^+(t) = N_N^+(t) - \left(\frac{k-1}{2}\right)$ and $N_D^-(t) = N_N^-(t) - \left(\frac{k-1}{2}\right)$, and this implies (5.27) and (5.28). □

Now we define the refined asymptotics

$$\widetilde{N}_N^+(t) = \frac{t}{4m} + \left(\frac{1}{4} + \frac{1}{4m}\right)\sqrt{t + \frac{1}{4}} + \frac{1}{24}\left(m - \frac{1}{m}\right) + \frac{1}{8} + \frac{1}{12m} \tag{5.30}$$

$$\widetilde{N}_N^-(t) = \frac{t}{4m} + \left(\frac{1}{4} - \frac{1}{4m}\right)\sqrt{t + \frac{1}{4}} + \frac{1}{24}\left(m - \frac{1}{m}\right) - \frac{1}{8} + \frac{1}{12m} \tag{5.31}$$

$$\widetilde{N}_D^+(t) = \frac{t}{4m} + \left(-\frac{1}{4} + \frac{1}{4m}\right)\sqrt{t + \frac{1}{4}} + \frac{1}{24}\left(m - \frac{1}{m}\right) - \frac{1}{8} + \frac{1}{12m} \tag{5.32}$$

$$\widetilde{N}_D^-(t) = \frac{t}{4m} + \left(-\frac{1}{4} - \frac{1}{4m}\right)\sqrt{t + \frac{1}{4}} + \frac{1}{24}\left(m - \frac{1}{m}\right) + \frac{1}{8} + \frac{1}{12m}. \tag{5.33}$$

Note that the coefficients of $\sqrt{t+\frac{1}{4}}$ may be explained because the two quarter circle boundary arcs have total length π, contributing $\frac{1}{4}$ to $\widetilde{N}_N^{\pm}$ and $-\frac{1}{4}$ to $\widetilde{N}_D^{\pm}$, and the equatorial boundary arc has length $\frac{\pi}{m}$, contributing $\frac{1}{4m}$ to $\widetilde{N}_N^{+}$ and $\widetilde{N}_D^{+}$ and $-\frac{1}{4m}$ to $\widetilde{N}_N^{-}$ and $\widetilde{N}_D^{-}$. For the constant term we note that the top angle $\frac{\pi}{m}$ always has the same type of boundary conditions on either side, so it contributes $\frac{1}{24}\left(m-\frac{1}{m}\right)$, while the bottom right angles have like boundary conditions for $\widetilde{N}_N^{+}$ and $\widetilde{N}_D^{-}$, so contribute $\frac{1}{8}$ to each, and opposite boundary conditions for $\widetilde{N}_N^{-}$ and $\widetilde{N}_D^{+}$ and so contribute $-\frac{1}{8}$ to each.

Theorem 5.6. *The analog of Theorem 5.3 holds.*

Proof. If suffices to understand the contribution of the parity dependent terms in Lemma 5.5. Suppose first that m is even. When we average the function

$$\phi(p) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \frac{m-p}{2m} & \text{if } p \text{ is odd} \end{cases}$$

over $1 \le p \le m$ we obtain

$$\frac{1}{m}\sum_{j=0}^{\frac{m-1}{2}} \frac{2j+1}{2m} = \frac{1}{2m^2}\left(\frac{m}{2}\right)^2 = \frac{1}{8}$$

so the average of $\phi(p) - \frac{1}{8}$ is $O(t^{-1})$.

Next suppose m is odd. This time we have to average the functions

$$\phi_0(p) = \frac{m-p}{4m}$$

when n is odd, and

$$\phi_1(p) = \frac{m-p}{4m} + \begin{cases} \frac{1}{4} & \text{if } p \text{ is odd} \\ -\frac{1}{4} & \text{if } p \text{ is even} \end{cases}$$

when n is even. Now the average value of ϕ_0 is $\frac{m(m-1)}{8m^2} = \frac{1}{8} - \frac{1}{8m}$, and the average value of ϕ_1 is $\frac{1}{8} - \frac{1}{8m} + \frac{1}{4m} = \frac{1}{8} + \frac{1}{8m}$ because there is exactly one more odd value of p than even values of p in $1 \le p \le m$. Thus when we average over n we get $\frac{1}{8}$. $\square$

6 Cone point singularities

The first example of a surface with cone point singularities is the flat projective plane obtained from the unit square by identifying both pairs of opposite edges with reversed orientation. This yields two singular points with cone angle π. This example was analyzed in [JS] where it was shown that

$$N(t) = \frac{1}{4} N_T(t) + \frac{1}{4} \pm \frac{1}{2} \tag{6.1}$$

where T is the 2×2 torus, and the choice of $\pm$ sign corresponds to the parity of $\left[\frac{t^{1/2}}{\pi}\right]$. Thus the choice

$$\widetilde{N}(t) = \frac{1}{4\pi} t + \frac{1}{4} \tag{6.2}$$

leads to the analog of Theorem 2.2. We note that the two cone points yield $2 \cdot 2\psi(\frac{\pi}{2}) = \frac{1}{4}$ for the constant in (6.2).

Our next example is the surface of a regular tetrahedron. This is discussed in detail in [GKS]. Let T be the torus associated to the lattice $\mathscr{L}$ generated by the vectors $(2, 0)$ and $(1, \sqrt{3})$. This is similar to the hexagonal torus discussed in section 2 but it is larger, and a fundamental domain consists of eight equilateral triangles. There is a two-fold covering of the tetrahedron by T (deleting singular points), and so the eigenfunctions on T have the form

$$e(k, j) = e^{2\pi i (ku + jv) \cdot x} \tag{6.3}$$

for $u = (\frac{1}{2}, \frac{\sqrt{3}}{6})$ and $v = (0, \frac{\sqrt{3}}{3})$, generators of the dual lattice $\mathscr{L}^*$, and $(k, j) \in \mathbb{Z}^2$, with eigenvalue

$$\frac{4\pi^2}{3}(j^2 + k^2 + jk), \tag{6.4}$$

while the eigenfunctions on the tetrahedron are

$$\cos 2\pi (ku + jv) \cdot x = \cos 2\pi \left(\frac{k}{2} x_1 + \left(\frac{\sqrt{3}}{3} j + \frac{\sqrt{3}}{6} k \right) x_2 \right). \tag{6.5}$$

Aside from $(k, j) = (0, 0)$, the two choices (k, j) and $(-k, -j)$ collapse to one choice in (6.5), so

$$N(t) = \frac{1}{2} N_T(t) + \frac{1}{2} \tag{6.6}$$

This leads to the choice

$$\widetilde{N}(t) = \frac{\sqrt{3}}{4\pi} t + \frac{1}{2} \tag{6.7}$$

and the analog of Theorem 2.2 holds. Note that the tetrahedron has four cone points with cone angles π, and $4 \cdot 2\psi(\frac{\pi}{2}) = \frac{1}{2}$.

Next we consider a half of the tetrahedron sliced so that two adjacent faces are cut by the perpendicular bisectors of their common edge. The boundary of the half-tetrahedron consists of the cut line together with a side edge of the face that is entirely in the surface (see Figure 6.1).

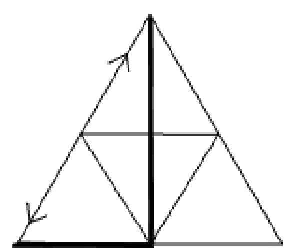


Figure 6.1: Half-tetrahedron with boundary line in dark.

If we impose Neumann boundary conditions then for generic choice of (k, j) we have eigenfunctions

$$\cos 2\pi(ku + jv) \cdot x + \cos 2\pi(-ku + (k + j)v) \cdot x, \tag{6.8}$$

while with Dirichlet boundary conditions we have

$$\cos 2\pi(ku + jv) \cdot x - \cos 2\pi(-ku + (k + j)v) \cdot x \tag{6.9}$$

corresponding to (k, j) and $(-k, k + j)$ in (6.5). However, in the nongeneric cases, if $k = 0$ then we have one function of the form (6.8) and none of the form (6.9) corresponding to a single function of the form (6.5), and similarly for $(2k, -k)$. Thus

$$N(t) = \frac{1}{4} N_T(t) + \frac{1}{2} \left(\left[\frac{\sqrt{3} t^{1/2}}{2\pi} \right] + \left[\frac{t^{1/2}}{2\pi} \right] \right) + \frac{3}{4} \tag{6.10}$$

for Neumann boundary conditions, and

$$N(t) = \frac{1}{4}N_T(t) - \frac{1}{2}\left(\left[\frac{\sqrt{3}t^{1/2}}{2\pi}\right] + \left[\frac{t^{1/2}}{2\pi}\right]\right) - \frac{1}{4} \tag{6.11}$$

for Dirichlet boundary conditions. Thus we choose

$$\widetilde{N}(t) = \frac{\sqrt{3}}{8\pi}t \pm \left(\frac{(\sqrt{3}+1)}{4\pi}t^{1/2}\right) + \frac{1}{4} \tag{6.12}$$

(+ for Neumann and − for Dirichlet) in order to obtain the analog of Theorem 2.2. Note that $\sqrt{3}+1$ is the length of the boundary. To explain the constant term we have to remember that angles at corners have to be measured in terms of the intrinsic geometry of the surface. Thus the boundary actually only has two corners, each of angle $\frac{\pi}{2}$, so these contribute $2 \cdot \frac{1}{16} = \frac{1}{8}$.

Our last set of examples are the glued lunes, where we identify the boundary arcs of a lune in the same orientation. Here we will be able to allow any positive integer m for the lune L_m, so the glued lune $\widetilde{L}_m$ will have two cone points with cone angle $\frac{2\pi}{m}$. Eigenfunctions on $\widetilde{L}_m$ are just spherical harmonics on the sphere satisfying rotation invariance under $z \mapsto e^{\frac{2\pi i}{m}}z$. As in section 5, we may identify a basis of the space $\mathbb{P}_N$ of homogeneous polynomials of degree N with this invariance as functions of the form $|z|^{2j}z^{mk}w^{\ell}$ or $|z|^{2j}\bar{z}^{mk}w^{\ell}$ with $2j + mk + \ell = N$. For any fixed j and k with $2j + mk \leq N$ there is a unique choice of ℓ, and there are two basis elements when $k > 0$ and one when $k = 0$. By considering the map $f \mapsto |z|^2 f$ from $\widetilde{\mathbb{P}}_{N-2}$ to $\widetilde{\mathbb{P}}_N$ we may compute the difference $\dim \widetilde{\mathbb{P}}_N - \dim \widetilde{\mathbb{P}}_{N-2}$ by counting the basis elements in $\widetilde{\mathbb{P}}_N$ corresponding to the choice $j = 0$, so analogous to Lemma 5.1 we have

$$\dim \widetilde{\mathbb{P}}_N - \dim \widetilde{\mathbb{P}}_{N-2} = 2\left[\frac{N}{m}\right] + 1 \tag{6.13}$$

Note that this is exactly the sum of (5.4) and (5.5), since the eigenspace for $\widetilde{L}_m$ is exactly the sum of the Neumann and Dirichlet eigenspaces of L_{2m}. In particular, to get $N(t)$ for $\widetilde{L}_m$ we simply need to add (5.8) and (5.9), so if $k \equiv p \mod m$ and $k^2 - k \leq t < k^2 + k$ we have

$$N(t) = \frac{k^2}{m} + \frac{p(m-p)}{m}. \tag{6.14}$$

Just as in (5.10) we choose

$$\widetilde{N}(t) = \frac{t}{m} + \frac{1}{6}(m - \frac{1}{m}) + \frac{1}{3m} \tag{6.15}$$

and obtain the analog of Theorem 5.3. The contribution of the two cone angles to the constant is $2 \cdot 2\psi(\frac{\pi}{m}) = \frac{1}{6}(m - \frac{1}{m})$.

7 Sorting by symmetry types

Suppose the surface has a finite group G of isometries. The group action then commutes with the Laplacian, so the eigenspaces are preserved, and we may sort the eigenfunctions by symmetry type, specifically the irreducible representations $\{\pi_j\}$ of G. This situation was discussed for a more general setting in [S], where it was suggested that asymptotically, the proportion of eigenfunctions of each symmetry type should be $\dfrac{d_j^2}{\#G}$, where $d_j = \dim \pi_j$ (it is well known that $\sum d_j^2 = \#G$). This was proposed as a heuristic idea that is not universally valid (there are simple counterexamples), but should hold if there is a fundamental domain for the group whose boundary is relatively small. In the case of surfaces there is usually a fundamental domain that is a polygon, so its boundary is one-dimensional. Here we will be able to prove a more refined statement for some of the surfaces we have examined: the square torus, the square, the hexagonal torus and the equilateral triangle. In the first two cases the group is the dihedral group D_4, and in the second two cases it is the dihedral group D_3.

The group D_4 has 8 elements that are generated by reflections in the diagonals of the square and the reflections in the horizontal or vertical bisectors. There are 4 1-dimensional representations $1 \pm \pm$, where the first $\pm$ indicates the symmetry type

$$u(Rx) = \pm u(x) \tag{7.1}$$

for diagonal reflections. and the second $\pm$ indicates the same symmetry equation where R is a horizontal or vertical reflection. There is also one 2-dimensional representation that we denote by 2. Write $N_{\pm\pm}(t)N$ and $N_2(t)$ for the counting function of all eigenvalues $\lambda \leq t$ that belong to each symmetry type. (Note that an individual eigenspace may contain eigenfunctions of more than one symmetry type, and for $N_2(t)$ we count each eigenfunction

basis element separately.)

For simplicity we choose a square of side length one, and this will represent the square torus if we identify opposite sides, or the square itself with either Neumann or Dirichlet boundary conditions throughout (we nix mixed boundary conditions because they do not respect the D_4 action). So in all three cases we have

$$N_{++}(t) + N_{+-}(t) + N_{-+}(t) + N_{--}(t) + N_2(t) = N(t), \tag{7.2}$$

where $N(t)$ is given by $N_T(t)$ for the torus, and $N_N(t)$ and $N_D(t)$ in Lemma 2.1 with $a = b = 1$.

A fundamental domain for the D_4 action is a right isosceles triangle of hypotenuse length $\frac{\sqrt{2}}{2}$ and the equal side lengths $\frac{1}{2}$ that makes up an eighth of the square. The key observation is that an eigenfunction of symmetry type $1 \pm \pm$, when restricted to the triangle, satisfies Dirichlet or Neumann boundary conditions as illustrated in Figure 7.1, and conversely every triangle eigenfunction extends by the appropriate reflections to an eigenfunction on the torus or square. Thus $N_{\pm}(t)$ is exactly the counting function for the triangle given by Lemma 2.3 and (3.22-3.30) with $a = \frac{1}{2}$. Then we can use (7.2) to compute $N_2(t)$.

Lemma 7.1. *(a) For the square torus we have*

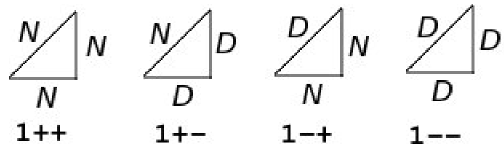
$$N_{++}(t) = \frac{1}{8}N_T(t) + \frac{1}{2}\left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) + \frac{3}{8} \tag{7.3}$$

$$N_{+-}(t) = \frac{1}{8}N_T(t) - \frac{1}{2}\left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) + \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) - \frac{1}{8} \tag{7.4}$$

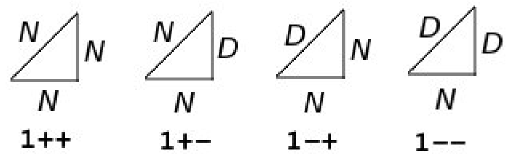
$$N_{-+}(t) = \frac{1}{8}N_T(t) + \frac{1}{2}\left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) - \frac{1}{8} \tag{7.5}$$

$$N_{--}(t) = \frac{1}{8}N_T(t) - \frac{1}{2}\left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) - \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) + \frac{3}{8} \tag{7.6}$$

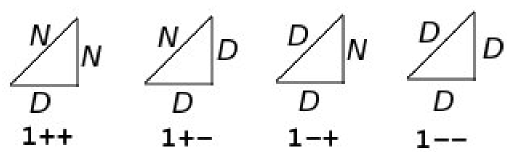
$$N_2(t) = \frac{1}{2}N_T(t) - \frac{1}{2} \tag{7.7}$$



(a) The square torus



(b) The square with Neumann boundary conditions



(c) The square with Dirichlet boundary conditions

Figure 7.1

(b) For the square with Neumann boundary conditions $N_{++}(t)$ is given by (7.3), $N_{-+}(t)$ is given by (7.5),

$$N_{+-}(t) = \frac{1}{2}N_{MM}(t) + \frac{1}{2}\left(\left[\frac{\sqrt{2}}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) \tag{7.8}$$

$$N_{--}(t) = \frac{1}{2}N_{MM}(t) - \frac{1}{2}\left(\left[\frac{\sqrt{2}}{4\pi}t^{1/2}\right] + \frac{1}{2}\right) \tag{7.9}$$

$$N_2(t) = \frac{3}{4}N_T(t) - N_{MM}(t) - \left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) - \frac{1}{4} \tag{7.10}$$

where N_{MM} is given by (3.10) with $a = \frac{1}{2}$.
(c) For the square with Dirichlet boundary conditions $N_{+-}(t)$ is given by

(7.4), $N_{--}(t)$ is given by (7.6),

$$N_{+-}(t) = \frac{1}{2}N_{MM}(t) + \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) \tag{7.11}$$

$$N_{-+}(t) = \frac{1}{2}N_{MM}(t) - \frac{1}{2}\left(\left[\frac{\sqrt{2}t^{1/2}}{4\pi}\right] + \frac{1}{2}\right) \tag{7.12}$$

$$N_2(t) = \frac{3}{4}N_T(t) - N_{MM}(t) - \left(\left[\frac{t^{1/2}}{2\pi}\right] + \frac{1}{2}\right) - \frac{1}{4} \tag{7.13}$$

Proof. Direct substitution of triangle formulas. □

We now define the refined asymptotics:
(a) For the square torus

$$\widetilde{N}_{++}(t) = \frac{1}{32\pi}t + \left(\frac{1}{4\pi} + \frac{\sqrt{2}}{8\pi}\right)t^{1/2} + \frac{3}{8} \tag{7.14}$$

$$\widetilde{N}_{+-}(t) = \frac{1}{32\pi}t + \left(-\frac{1}{4\pi} + \frac{\sqrt{2}}{8\pi}\right)t^{1/2} - \frac{1}{8} \tag{7.15}$$

$$\widetilde{N}_{-+}(t) = \frac{1}{32\pi}t + \left(\frac{1}{4\pi} - \frac{\sqrt{2}}{8\pi}\right)t^{1/2} - \frac{1}{8} \tag{7.16}$$

$$\widetilde{N}_{--}(t) = \frac{1}{32\pi}t + \left(-\frac{1}{4\pi} - \frac{\sqrt{2}}{8\pi}\right)t^{1/2} + \frac{3}{8} \tag{7.17}$$

$$\widetilde{N}_2(t) = \frac{1}{8\pi}t - \frac{1}{2}; \tag{7.18}$$

(b) For the square with Neumann boundary conditions $\widetilde{N}_{++}$ is given by (7.14), $\widetilde{N}_{-+}$ is given by (7.16),

$$\widetilde{N}_{+-}(t) = \frac{1}{32\pi}t + \frac{\sqrt{2}}{8\pi}t^{1/2} \tag{7.19}$$

$$\widetilde{N}_{--}(t) = \frac{1}{32\pi}t - \frac{\sqrt{2}}{8\pi}t^{1/2} \tag{7.20}$$

$$\widetilde{N}_2(t) = \frac{1}{8\pi}t - \frac{1}{2\pi}t^{1/2} - \frac{1}{4}; \tag{7.21}$$

(c) For the square with Dirichlet boundary conditions $\widetilde{N}_{+-}(t)$ is given by the right side of (7.15), $\widetilde{N}_{--}(t)$ is given by the right side of (7.17), $\widetilde{N}_{++}(t)$ is given by (7.19), $\widetilde{N}_{-+}(t)$ is given by (7.20),

$$\widetilde{N}_2(t) = \frac{1}{8\pi}t + \frac{1}{2\pi}t^{1/2} - \frac{1}{4}. \tag{7.22}$$

Then the analog of Theorem 2.2 holds.

Next we consider the case of the hexagonal torus associated to the hexagon in Figure 2.1. The group D_3 of order six is generated by the three reflections in the three bisectors. It has two 1-dimensional representations denoted $1\pm$ according to the symmetry (7.1) with respect to all three reflections, and a 2-dimensional representation denoted 2. The analog of (7.2) is

$$N_+(t) + N_-(t) + N_2(t) = N_T(t), \tag{7.23}$$

and we use this to solve for $N_2(t)$. Since $1+$ eigenfunctions may be identified with Neumann eigenfunctions on the equilateral triangle fundamental domain (shaded in Figure 2.1), and $1-$ eigenfunctions with Dirichlet eigenfunctions, we have $N_+(t)$ given by (2.22) and $N_-(t)$ by (2.23). This leads to

$$\widetilde{N}_+(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{4}t + \frac{3}{4\pi}t^{1/2} + \frac{1}{3} \tag{7.24}$$

$$\widetilde{N}_-(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{4}t - \frac{3}{4\pi}t^{1/2} + \frac{1}{3} \tag{7.25}$$

$$\widetilde{N}_2(t) = \frac{\sqrt{3}}{4\pi}t - \frac{2}{3} \tag{7.26}$$

Finally we consider the equilateral triangle with either Neumann or Dirichlet boundary conditions. A fundamental domain for the action of D_3 is a $30^\circ - 60^\circ - 90^\circ$ triangle equal to a sixth of the equilateral triangle. Again, the eigenfunctions on the equilateral triangle with $1\pm$ symmetry correspond to eigenfunctions on the fundamental domain with boundary conditions shown in Figure 7.2.

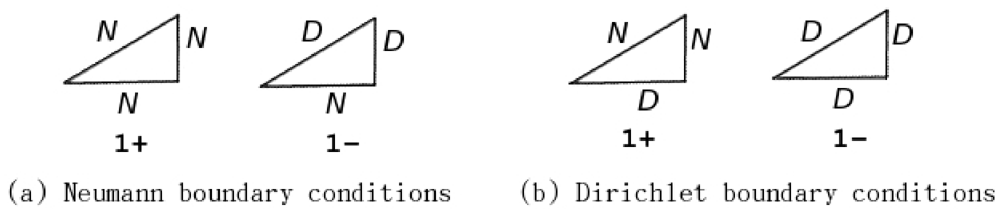


Figure 7.2

Thus for Neumann boundary conditions, $N_T(t)$ is given by (2.26) and $N_-(t)$ is given by (3.32), while for Dirichlet boundary conditions $N_+(t)$ is given by (3.31) and $N_-(t)$ is given by (2.27). (Here we have to rescale because the triangles are smaller by a factor of $\frac{1}{\sqrt{3}}$.) This leads to the following choices:

(a) For Neumann boundary conditions

$$\widetilde{N}_+(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{24}t + \frac{\sqrt{3}+1}{8\pi}t^{1/2} + \frac{5}{12} \tag{7.27}$$

$$\widetilde{N}_-(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{24}t + \frac{-\sqrt{3}+1}{8\pi}t^{1/2} - \frac{1}{12} \tag{7.28}$$

$$\widetilde{N}_2(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{6}t - \frac{1}{4\pi}t^{1/2} - \frac{1}{3}; \tag{7.29}$$

(b) For Dirichlet boundary conditions

$$\widetilde{N}_+(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{24}t + \frac{\sqrt{3}-1}{8\pi}t^{1/2} - \frac{1}{12} \tag{7.30}$$

$$\widetilde{N}_-(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{24}t - \frac{\sqrt{3}+1}{8\pi}t^{1/2} + \frac{5}{12} \tag{7.31}$$

$$\widetilde{N}_2(t) = \frac{1}{4\pi}\frac{\sqrt{3}}{6}t + \frac{1}{4\pi}t^{1/2} - \frac{1}{3}. \tag{7.32}$$

Again the analog of Theorem 2.2 holds.

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